Percolation

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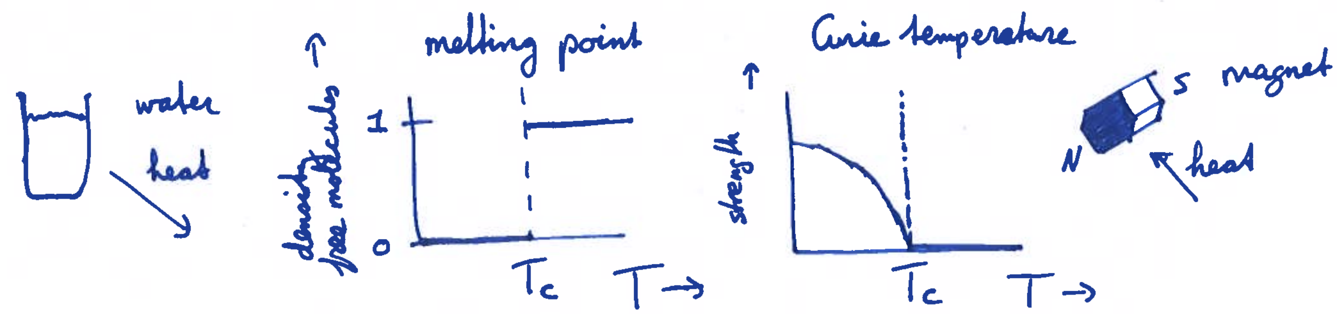
10×2 hrs, written final exam, midterm tbd.



- Nature is often continuous.
- When particle systems behave discontinuously, we speak of <u>phase transition</u>.
- Can phase transitions be modelled mathematically?
- What happens at or around the <u>critical point</u>?
  → Is the phase transition <u>continuous</u>?
- Can we <u>classify</u> phase transitions?

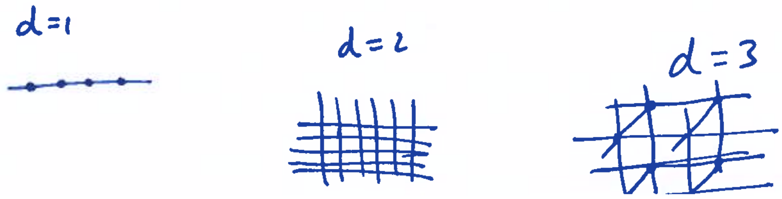
<u>Statistical mechanics</u> (stat. mech.) studies these questions.
<u>Percolation</u> is the simplest model in stat. mech. (for probabilists).

Chapter 1. Square lattice and self-avoiding walk (SAW).

Particles are discrete: we need a discretisation of Euclidean space.

Definition 1.1. ~~The square lattice in dimension~~ Consider the standard basis $(e_i)_{1\leq i\leq d}$ of $\mathbb{R}^d$ in dimension $d\geq 1$. The <u>square lattice</u> is the graph $(\mathbb{Z}^d, E)$ with vertex set $\mathbb{Z}^d \subseteq \mathbb{R}^d$, and such that the neighbours of any $x\in\mathbb{Z}^d$ are

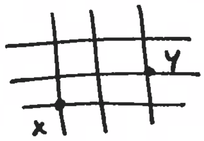
$$x \pm e_i .$$



etc.

Remark 1.2. The square lattice is <u>vertex-transitive</u>.

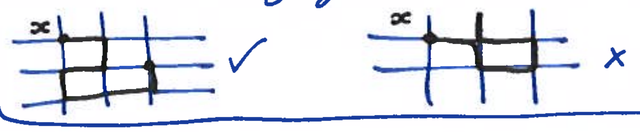
$\boxed{?}$ $\forall x,y \in \mathbb{Z}^d$, $\exists \varphi$ graph automorphism, $\varphi(x)=y$



$\varphi =$ "shift 2R, 1U".

To get a feeling for the lattice, we define a simple model.

Definition 1.3. A <u>self-avoiding walk</u> of length n is a sequence of $n+1$ <u>unique</u> vertices $(a_k)_{0\le k\le n}$ such that $a_k \sim a_{k+1}$ for all $0 \le k < n$, that is, such that $a_k a_{k+1} \in \mathbb{E}$.
Let c_n denote # such walks starting from $x \in \mathbb{Z}^d$.



Asymptotics of $(c_n)_{n\ge 0}$.

- Clearly $d^n \le c_n \le (2d)^n$, so $d \le \sqrt[n]{c_n} \le 2d$.
- Does $\sqrt[n]{c_n}$ converge? $\rightsquigarrow$ yes, connective constant
- Does c_{n+1}/c_n converge? $\rightsquigarrow$ open problem

$= \mathbb{E}_n[\#\text{ free vertices adjacent to endpoint}]$

↑ Uniform probability measure on SAW of length n from x.

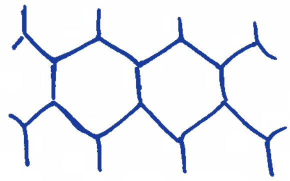
Claim 1.4. $c_{a+b} \le c_a \times c_b$ „subadditivity"

Proof: exercise.

Theorem 1.5. $\sqrt[n]{c_n}$ converges.

Proof: exercise. Hint: prove that $\sqrt[k]{c_k} \geq \limsup_n \sqrt[n]{c_n}$.

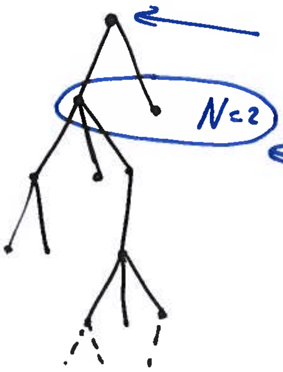
SAW:

- Many open problems remain.
- It is expected that $\mu := \lim_n \sqrt[n]{c_n}$ cannot be calculated, except in special cases.
- On the hexagonal lattice,  , $\mu = \sqrt{2+\sqrt{2}}$.

(Duminil-Copin, Smirnov 2012, (nonrig.: Nienhuis 1982)).

Proof uses discrete complex analysis, not expected to work for other graphs.

Chapter 2. Review of Galton-Watson trees.



- Start with one individual
- Produces random number N of children
- Each child produces children with the same distribution, i.i.d.
- Iterated indefinitely or until no new children are born.

Useful for population dynamics, but also in stat mech.

Definition 2.1. Galton-Watson tree. Let N denote a probability distribution on $\mathbb{Z}_{\geq 0}$. Let $(N_{a,b})_{a,b\geq 0}$ denote i.i.d. copies. The GW-tree with offspring N is defined by

- $Z_0 = 1$
- $Z_{n+1} = N_{n,1} + \dots + N_{n,Z_n} \quad \forall n \geq 0.$

Note: if $Z_n = 0$, then $Z_{n'} = 0 \quad \forall n' \geq n$.

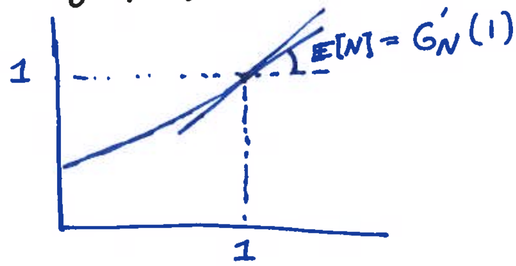
Note: $\{\sum_n Z_n = \infty\} = \{Z_n > 0 \ \forall n\} =: \{\text{survival}\}$,

$\{\sum_n Z_n < \infty\} = \{\lim_{n\to\infty} Z_n = 0\} =: \{\text{extinction}\}$.

Definition 2.2. If N is a random variable, then its generating function is

$$G_N(x) := \mathbb{E}[x^N].$$

~~Proposition 2~~ The generating function allows us to understand phase transition for G.W.-trees. Note that if N is an offspring distribution, then G_N satisfies the following properties:



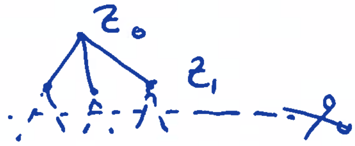
- $G_N(x) \geq 0 \quad \forall x \geq 0$
- $G_N(1) = 1$
- G_N is increasing on $[0,\infty)$ and also convex
- $G_N'(1) = \mathbb{E}[N]$ (if well-defined)
- If $N \leq C \in \mathbb{R}$ a.s., then G_N is a polynomial.

Proposition 2.3. Let $f : \mathbb{R} \to \mathbb{R}$ denote an increasing continuous function. Fix $\lambda_0 \in \mathbb{R}$, and define $\lambda_{k+1} := f(\lambda_k)$. If $(\lambda_k)_{k\geq 0}$ is non-decreasing, then $\lim_{k\to\infty} \lambda_k$ equals the smallest fixed point of f $\geq \lambda_0$ if it exists, or ∞ otherwise.

Proof. Exercise.

Theorem 2.4. Consider a G.W.-tree with offspring N. Then $\mathbb{P}(\text{extinction})$ equals the smallest fixed point ≥ 1 of G_N. In particular, $\mathbb{P}(\text{extinction}) = 1$ if and only if $\mathbb{E}[N] \leq 1$.

Proof. First, we cut the tree.



Define $E_k := \{Z_k = 0\} = \{\text{tree extinct with } k \text{ generations}\}$.

Note $E_k \subseteq E_{k+1} \quad \forall k$ and $\bigcup_k E_k = \{\text{extinction}\}$.

Clearly $Z_0 = 1$ and therefore $\mathbb{P}(E_0) = 0$.

Now $\mathbb{P}(E_{k+1}) = \mathbb{P}\left(\begin{array}{l} N \text{ independent G.W.-trees go extinct} \\ \text{within } k \text{ generations} \end{array}\right)$

$$= G_N(\mathbb{P}(E_k)).$$

Thus, setting $\lambda_0 = 0$; $\lambda_{k+1} = G_N(\lambda_k)$, and observing that the conditions of Proposition 2.3 are satisfied, we conclude. $\square$

Galton-Watson trees are useful for proving that random variables have exponential decay. We introduce this concept first.

Definition 2.5.

- A sequence $(a_n)_{n \geq 0}$ has exponential decay if
$$\exists \alpha, C > 0, \quad |a_n| \leq C e^{-\alpha n} \quad \forall n.$$
- A random variable X has exponential decay if
$$\exists \alpha > 0, \quad \mathbb{E}[e^{\alpha |X|}] < \infty.$$

Remark 2.6. The Markov inequality gives:

$$\mathbb{P}[|X| \geq \lambda] \leq e^{-\alpha \lambda} \mathbb{E}[e^{\alpha |X|}] \quad \forall \lambda \geq 0.$$

Theorem 2.7. Consider a G.W.-tree with offspring $N \le C$ a.s., where C is a finite constant. Then for all $\alpha > 0$,

$$\mathbb{E}\left[e^{\alpha \sum_n Z_n}\right]$$

is the smallest ~~sol~~ fixed point ≥ 1 of

$$x \mapsto e^\alpha G_N(x),$$

or ∞ if this does not exist. In particular, if $\mathbb{E}[N] < 1$, then $\sum_n Z_n$ has exponential decay.

Proof. First, we cut the tree.

$$X_k := Z_0 + \cdots + Z_k \qquad X_k \underset{k\to\infty}{\uparrow} X_\infty := \sum_n Z_n.$$

Since $N \le C$ a.s., G_N is a polynomial, and $X_k \le 1 + C + \cdots + C^k < \infty$ a.s. Thus

$$\lambda_k := \mathbb{E}\left[e^{\alpha X_k}\right] < \infty.$$

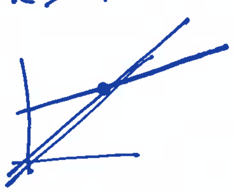
Want $\lambda_\infty := \lim_{k\to\infty} \lambda_k = \mathbb{E}\left[e^{\alpha X_\infty}\right] = \mathbb{E}\left[e^{\alpha \sum_n Z_n}\right]$.

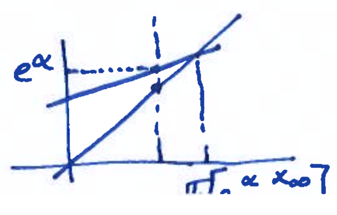
Since $X_0 = Z_0 = 1$, $\lambda_0 = e^\alpha$.

Now $\lambda_{k+1} = e^\alpha\, \mathbb{E}\left[\begin{matrix}\text{Product of "}\mathbb{E}e^{\alpha X_{\cancel{\infty} k}}\text{" for } N\\ \text{independent G.W.-trees}\end{matrix}\right] = \cancel{\mathbb{E}}\, e^\alpha G_N(\lambda_k)$

Note also that $\lambda_0 = e^\alpha G_N(\lambda_{-1})$ for $\lambda_{-1} := 1$.

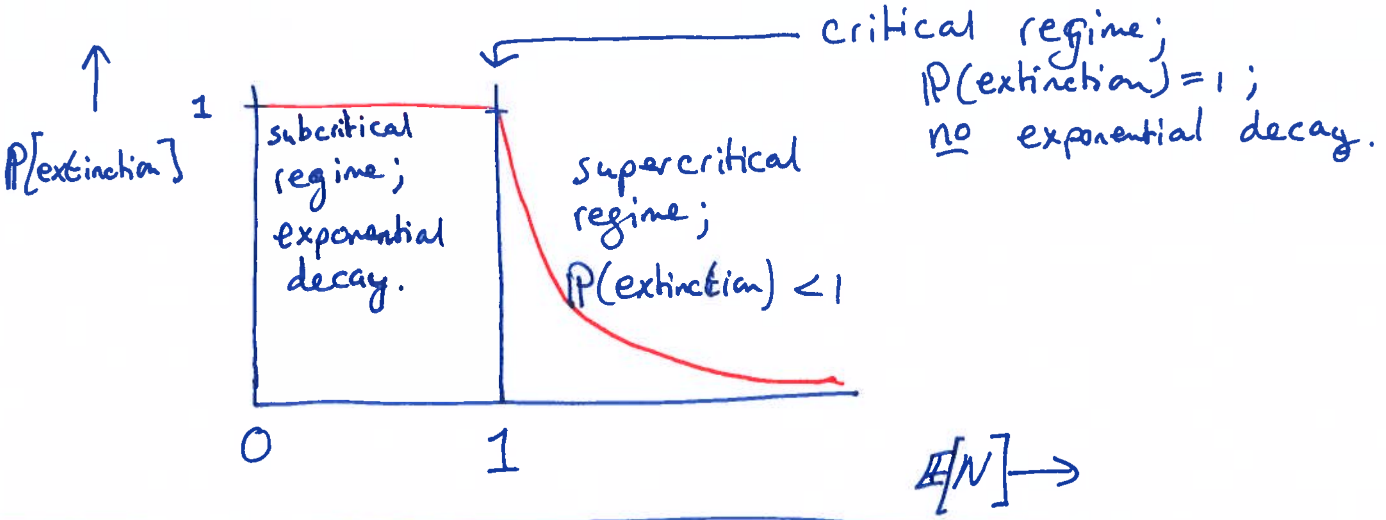
Since $(\lambda_k)_{k \ge -1}$ and G_N are non-decreasing, we conclude.





$\square$

Phase diagram 2.8. The phase diagram of the Galton-Watson tree. Suppose ~~E~~ $N \leq C < \infty$ a.s.



Chapter 3. Percolation definition and first results.

Definition 3.1. Let $G = (V, E)$ denote any graph, and let $p \in [0,1]$ denote a parameter. Bernoulli percolation with parameter p on G is:

- The sample space $\Omega = \{0,1\}^E$,
- The σ-algebra $\mathcal{F}$ generated by $(\{\omega_{xy} = 1\})_{xy \in E}$,
- The product measure $\mathbb{P}_p = P_p^{\otimes E}$, where

$$P_p = p\,\delta_1 + (1-p)\,\delta_0 .$$

Elements $\omega \in \Omega$ are called configurations. An edge $xy \in E$ is <u>open</u> if $\omega_{xy} = 1$ and <u>closed</u> if $\omega_{xy} = 0$. We also identify ω with the random ~~set~~ subgraph $\{\omega = 1\} \subseteq E$.

Notation 3.2. Let $G=(V,E)$ denote a graph and ω a configuration. In other words, (V,ω) is also a graph. Write:

- $x \longleftrightarrow y$ if $x, y \in V$ are connected in (V,ω),
- C_x for the connected component of x in (V,ω),
- $x \longleftrightarrow \infty$ for $|C_x| = \infty$,
- $\mathcal{E}_\infty := \bigcup_{x\in V} \{x \longleftrightarrow \infty\} \subseteq \Omega$,
- $\theta(p) := \cancel{\mathbb{P}_p(\mathcal{E}_\infty)}\ \mathbb{P}_p(x \longleftrightarrow \infty)$ using vertex-transitivity.
- $p_c := \sup\{p \in [0,1] : \theta(p) = 0\}$.

Exercise 3.3. ~~Prove that $\mathcal{E}_\infty \in \mathcal{F}$.~~
(i) Prove that $\{x \leftrightarrow y\} \in \mathcal{F}$.
(ii) Prove that $\{x \leftrightarrow \infty\} \in \mathcal{F}$.
(iii) Prove that $\mathcal{E}_\infty \in \mathcal{F}$.

Exercise 3.4. Percolation on regular trees.
Let $T_d = (V_d, E_d)$ denote the tree in which each vertex has exactly $d \geq 3$ neighbours.

(i) What is the distribution of $|C_x|$?

(ii) What is p_c?

(iii) Describe $\theta(p)$, $\begin{cases} \text{below } p_c, \\ \text{at } p_c, \\ \text{above } p_c. \end{cases}$

(iv) How does $|C_x|$ behave in each regime?

Trees are a nice playground, because each branch is independent. Other graphs are harder to deal with.