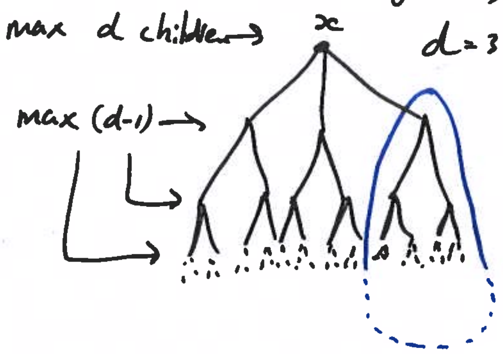
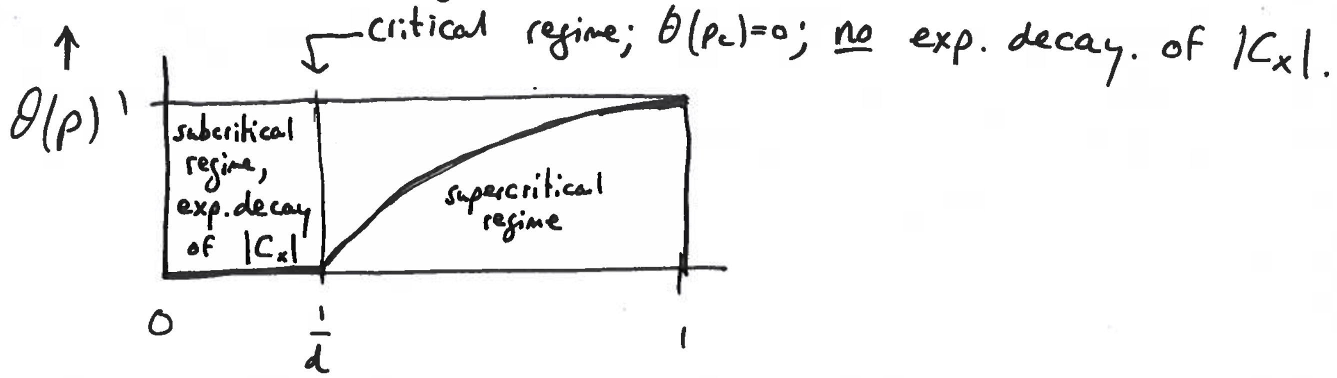
Solution to Exercise 3.4.

C_x has the structure of a Galton-Watson tree, except that the offspring distribution from $x \in V_d$ is different. ~~One can make thes same calculations as before, exct~~ Two strategies:



→ same formalism as before, with a different offspring distribution for the first generation.

→ ~~Each~~ Random number of "true" Galton-Watson trees.



Next, we prove phase transition on $(\mathbb{Z}^d)_{d\geq 2}$.

Clearly there is ~~on~~ no phase transition on $\mathbb{Z}^d$.

Theorem 3.5. In dimension d, ~~td~~ we have, (on $\mathbb{Z}^d$),

$$\theta(p) = 0 \qquad \forall\, p < \tfrac{1}{2d}.$$

Proof. Let SAW_n denote the set of self-avoiding walks of length n, and starting from $x \in \mathbb{Z}^d$. Then

$$|SAW_n| = c_n \le (2d)^n.$$

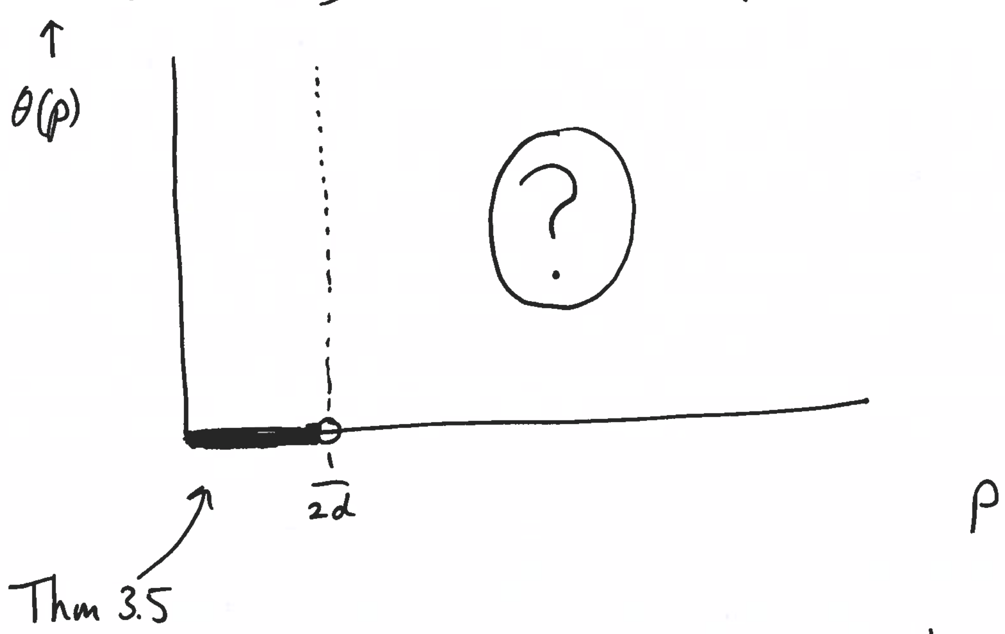
If $x \leftrightarrow \infty$, then ~~we~~ $\exists\, w \in SAW_n$ which is open for ω.

Thus,

$$\begin{aligned}
\mathbb{P}_p(x \leftrightarrow \infty) &\le \mathbb{P}_p\left(\exists \text{ open } w \in SAW_n\right) \\
&\overset{\text{union bound}}{\le} \sum_{w \in SAW_n} \mathbb{P}_p(w \text{ is } \omega\text{-open}) \\
&= |SAW_n|\, p^n \\
&\le \cancel{p}\ (2dp)^n \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

$\square$

Phase diagram of Bernoulli percolation on ~~the~~ $\mathbb{Z}^d$.



"This is what research looks like"

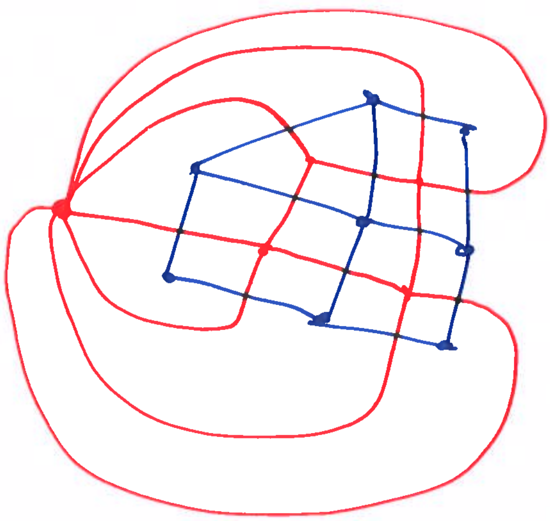
Definition 3.6. Always write
$$\Lambda_n := (-n,n)^d \cap \mathbb{Z}^d;$$
this is called the <u>box of size n</u>. Also write
$$\partial\Lambda_n := \Lambda_{n+1} \setminus \Lambda_n,$$
the <u>boundary of the box</u>.

Exercise 3.7. Show that in dimension d, for $p < \frac{1}{2d}$,
$$\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n)$$
has exponential decay in n.

Theorem 3.8. In dimension $\underline{d \geq 2}$ we have
$$\theta(p) > 0 \qquad \forall p > 1 - \varepsilon,$$
for $\varepsilon > 0$ small.

Definition 3.9. Let $G = (V,E)$ denote a planar graph, with F denoting the set of faces.

The <u>dual graph</u> is the graph $G^* = (V^*, E^*)$ such that $V^* = F$, and such that E^* connects adjacent faces. This graph is also planar, and $G^{**} = G$.



For $xy \in E$, let $xy^* \in E^*$ denote the corresponding <u>dual edge</u>.

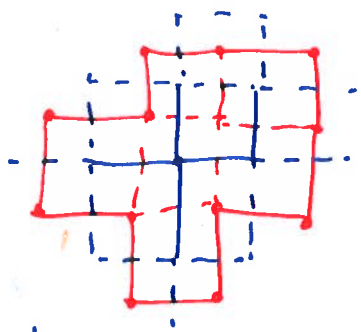
For $\omega \subseteq E$, let ω^* denote the set

$$\omega^* = \{ xy^* \in E^* : xy \boxed{\notin} \omega \}.$$

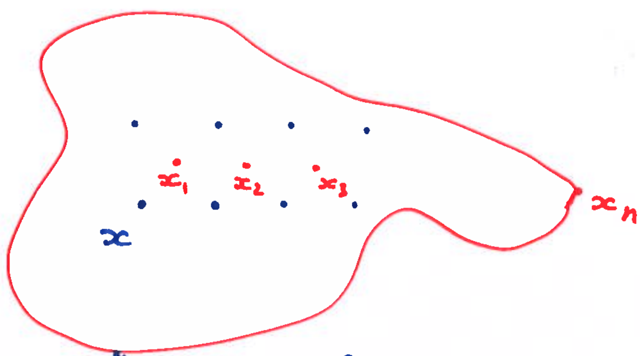
This is called the <u>dual</u> of ω.

Proof of Thm 3.8.

We focus on $\mathbb{Z}^2$, which is planar. In $\mathbb{P}_p$, ω^* has the distribution of $1-p$ percolation. If C_x is finite, then there is a dual open circuit surrounding x.



Let x_n denote the face on the north-west of $x + (n,0)$. Then



$$\{C_x \text{ is finite}\} = \bigcup_{n=1}^{\infty} \{\exists \text{ open dual } \underbrace{\text{circuit around } x \text{ through } x_n}_{\text{Such a circuit has length at least } 2n}\}.$$

$$\subseteq \bigcup_{n=1}^{\infty} \{\exists \text{ S.A.W. of length} \ge 2n \text{ from } x_n \text{ which is dual-open}\}.$$

Union bound:

$$\mathbb{P}_p(C_x \text{ is finite}) \le \sum_{n=1}^{\infty} \overbrace{\cancel{C_{2n}(1-p)^{2n}}}^{\sum_{k=2n}^{\infty} C_k (1-p)^k} \cancel{\le \sum} \le \sum_{n=1}^{\infty} \overbrace{\cancel{(4\varepsilon)^n}}^{\sum_{k=2n}^{\infty} (4\varepsilon)^k} < 1$$

for ε sufficiently small.

For $d > 2$, note that

$$\{C_x \text{ is infinite}\} \subseteq \{x \longleftrightarrow \infty \text{ using only edges in a 2D plane}\},$$

and therefore

$$\mathbb{P}_p(C_x \text{ is finite}) \le \sum_{n=1}^{\infty} (4\varepsilon)^n$$

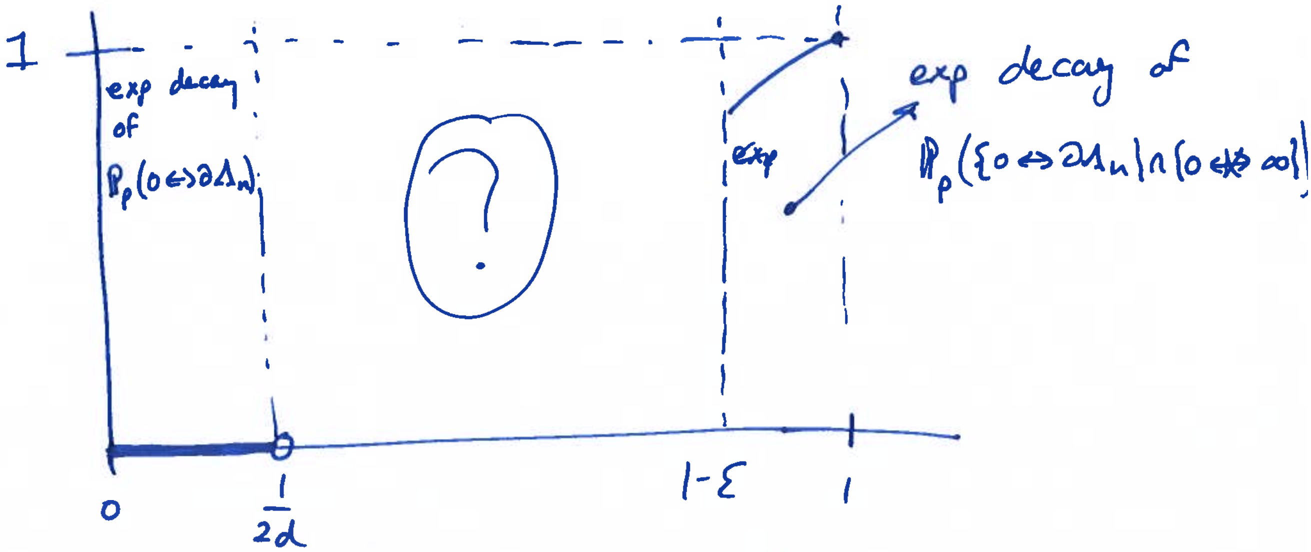
also. $\square$

Exercise 3.10. Without giving a full proof, argue that for $p > 1-\varepsilon$,

$$\mathbb{P}_p\left(\{0 \longleftrightarrow \partial\Lambda_n\} \cap \{0 \not\longleftrightarrow \infty\}\right)$$

is small when n is large.

Phase diagram of percolation on $\mathbb{Z}^d$.



Thm 3.5

Exercise 3.7

Thm 3.8.

Exercise 3.10

Definition 3.11. We say that ω is below ω' if $\omega_{xy} \leq \omega'_{xy}$ for all $xy \in E$. As sets, this is equivalent to $\omega \subseteq \omega'$.

Theorem 3.12. Let $p < p'$. There exists a <u>coupling</u> between $\mathbb{P}_p$ and $\mathbb{P}_{p'}$ such that almost surely,

$$\omega \subseteq \omega'.$$

In other words, there exists a measure P on $\{0,1\}^E \times \{0,1\}^E$ with $\mathbb{P}_p$ and $\mathbb{P}_{p'}$ as marginals, and such that

$$P(\omega \subseteq \omega') = 1.$$

Proof. Let $(U_e)_{e \in E}$ denote an i.i.d. family of random variables with distribution $U([0,1])$. Define

$$\omega_e = \mathbb{1}(U_e \leq p);$$

$$\omega'_e = \mathbb{1}(U_e \leq p').$$

This measure clearly satisfies the properties. $\square$.

Remarks. Recall that

$$p_c := \sup\{p \in [0,1] : \theta(p) = 0\}.$$

We want $\mathbb{P}_p(0 \leftrightarrow \infty) > 0 \quad \forall p > p_c$.

Suppose that $\omega \leq \omega'$. Then

$$0 \overset{\omega}{\longleftrightarrow} \infty \implies 0 \overset{\omega'}{\longleftrightarrow} \infty.$$

In particular,

$$\mathbb{P}_p(0 \leftrightarrow \infty) = P(0 \overset{\omega}{\leftrightarrow} \infty) \leq P(0 \overset{\omega'}{\leftrightarrow} \infty) = \mathbb{P}_{p'}(0 \leftrightarrow \infty).$$

Thus, θ is increasing in p.

Theorem 3.13. Θ is increasing in p.

Exercise 3.14. Show that $p_c(\mathbb{Z}^d) \leq \frac{3}{4}$ for $d \geq 2$.

Exercise 3.15. Show that

$$p \mapsto \mathbb{P}_p(a \leftrightarrow b \quad \text{and} \quad c \leftrightarrow d)$$

is increasing in p. What is a natural class of events A such that

$$p \longmapsto \mathbb{P}_p(A)$$

is increasing?

Definition 3.16 In site percolation, the vertices are open or closed instead of the edges.

Exercise 3.17. Show that

$$0 < p_c(\mathbb{Z}^d, \text{site}) < 1. \qquad \boxed{\text{HARD}}$$

Exercise 3.18. Show that edge percolation corresponds to site percolation on a modified graph.

Exercise 3.19. Consider an infinite vertex-transitive graph of degree Δ. Show that

$$p_c(\text{bond}) \le p_c(\text{site}) \le 1 - (1 - p_c(\text{bond}))^{\Delta}.$$

$\boxed{\text{VERY HARD}}$

Exercise 3.20. Assume the setting of the previous exercise. Show that for $p < \frac{1}{\Delta}$, $|C_x|$ has exponential decay in the measure $\mathbb{P}_p$.

$\boxed{\text{VERY HARD}}$

Chapter 4. Increasing events.

Definition 4.1. An <u>increasing function</u> is a function $f : \Omega = \{0,1\}^E \longrightarrow \mathbb{R}$ such that

$$f(\omega) \le f(\omega') \qquad \forall \; \omega \le \omega'.$$

An event A is <u>increasing</u> if $\mathbb{1}_A$ is increasing as a function. This is equivalent to asking that

$$\omega \in A \implies \omega' \in A \qquad \forall \; \omega \le \omega'.$$

An event A is ~~inc~~ <u>decreasing</u> if A^c is increasing.

Proposition 4.2. If f is increasing, then

$$p \mapsto \mathbb{E} f$$

is increasing.

Proof. Apply Theorem 3.12. $\square$.

Examples of increasing events:

- the edge e = xy is open,
- $x \longleftrightarrow y$,
- $x \longleftrightarrow \infty$,
- the number of open edges in Λ_n exceeds K,
- $f \geq C \in \mathbb{R}$, where f is an increasing function.

Exercise 4.3.

Show that increasing events are stable under taking unions and intersections. Note that $\mathbb{1}_A \mathbb{1}_B = \mathbb{1}_{A \cap B}$. Are increasing functions stable under multiplication?

We now present the fundamental inequality in percolation theory.

Theorem 4.4. The Harris inequality.

If A, B are increasing events, then

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\,\mathbb{P}_p(B).$$

More generally, if f and g are increasing functions,

$$\mathbb{E}_p(fg) \geq \mathbb{E}_p(f)\,\mathbb{E}_p(g).$$

Remarks (i) Also called FKG (Fortuin-Kasteleyn-Ginibre) inequality, after the authors who generalised the Harris inequality.

(ii) The inequality does not use at all the graph structure. We may see it as an inequality on functions on "random bits".

Proof. Let $(e_i)_{i\geq 1}$ denote an enumeration of $\mathbb{E}$. Write $\omega_i := \omega(e_i)$. Let $\mathcal{H}_n$ denote the statement

$\mathcal{H}_n$: "If f, g are increasing and $\mathcal{F}_n = \sigma(\omega_i : 1\leq i\leq n)$-measurable, then $\mathbb{E}_p(fg) \geq \mathbb{E}_p(f)\mathbb{E}_p(g)$."

- Focus on $n=1$. Then $f, g : \{0,1\} \to \mathbb{R}$, and $f(0) = g(0)$ w.l.o.g.

$$\mathbb{E}_p(fg) - \mathbb{E}_p(f)\mathbb{E}_p(g) = p f(1) g(1) - p^2 f(1) g(1) =$$
$$= p(1-p) f(1) g(1) \geq 0,$$

hence $\mathcal{H}_1$.

- Assume $\mathcal{H}_{n-1}$, focus on $\mathcal{H}_n$. Let $\mathbb{P}_{\omega_n}$ denote the law of ω_n.

(A) $\mathbb{E}_p[f \mid \mathcal{F}_{n-1}](\omega_1, \dots, \omega_{n-1}) = \mathbb{E}_{\omega_n}[f(\omega_1, \dots, \omega_{n-1}, \cdot)]$.

(B) $\mathbb{E}_{\omega_n}[f(\omega_1, \dots, \omega_{n-1}, \cdot) g(\text{—})] \overset{\mathcal{H}_1}{\geq} \mathbb{E}_{\omega_n}[f(\text{—})]\mathbb{E}_{\omega_n}[g(\text{—})]$.

In particular,

$$\mathbb{E}_p \mathbb{E}_p[fg \mid \mathcal{F}_{n-1}] \overset{(A)}{=} \mathbb{E}_p \mathbb{E}_{\omega_n}[f(\omega_1, \dots, \omega_{n-1}, \cdot) g(\text{—})]$$

$$\overset{(B)}{\geqslant} \mathbb{E}_p\Big[\underbrace{\mathbb{E}_{\omega_n}[(f(\text{—}))]}_{\text{increasing in } \omega_1, \dots, \omega_{n-1}} \underbrace{\mathbb{E}_{\omega_n}[g(\text{—})]}\Big]$$

$$\overset{\mathcal{H}_{n-1}}{\geqslant} \mathbb{E}_p\big[\mathbb{E}_{\omega_n}[f(\text{—})]\big] \mathbb{E}_p\big[\mathbb{E}_{\omega_n}[g(\text{—})]\big]$$

$$\overset{(A)}{=} \mathbb{E}_p f \, \mathbb{E}_p g.$$

This completes the proof for $\mathcal{H}_n$. What if f, g depend on infinitely many edges?

Let $f_n := \mathbb{E}_p(f \mid \mathcal{F}_n)$; $g_n := \mathbb{E}_p(g \mid \mathcal{F}_n)$.

Then $f = \lim f_n \quad g = \lim g_n \quad fg = \lim f_n g_n.$

Have

$$\mathbb{E}_p f_n g_n \geqslant \mathbb{E}_p f_n \, \mathbb{E}_p g_n.$$

DCT yields the answer. $\square$