Exercise 4.5.

Prove the Harris inequality for <u>decreasing</u> events.

Prove that $\mathbb{P}(A \cap B) \le \mathbb{P}(A)\,\mathbb{P}(B)$ whenever A is increasing and B decreasing.

Exercise 4.6 (Square-root trick).

Let $A_1, \dots, A_n$ denote increasing events.

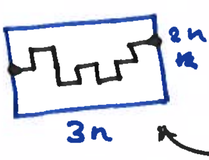
Prove that

$$\max_k \mathbb{P}(A_k) \geq 1 - \sqrt[n]{1 - \mathbb{P}(A_1 \cup \dots \cup A_n)}$$

In particular, if $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \dots = \mathbb{P}(A_n)$, then this gives a lower bound on all these probabilities.

Note that if $\mathbb{P}(A_1 \cup \dots \cup \dots \cup A_n) \approx 1$, then this means that $\max_k \mathbb{P}(A_k) \approx 1$.

Exercise 4.7. Consider dimension $d=2$. Fix p.

Let $a_n := \mathbb{P}_p\Big($ 

$\Big)$ the $3n \times 2n$ rectangle has an open horizontal crossing.

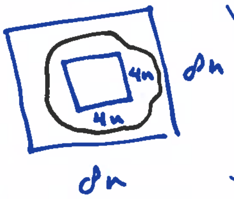
Suppose that $a_n \geq \varepsilon > 0$ $\forall n$, where ε is constant.

(i) Show that $\exists \varepsilon' > 0$ such that

$\mathbb{P}_p\Big($ 

$\Big) \geq \varepsilon' \qquad \forall n.$

(ii) Show that $\exists \varepsilon'' > 0$ such that

$\mathbb{P}_p\Big($ 

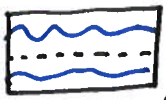
$\Big) \geq \varepsilon'' \qquad \forall n$

(iii) Show that, almost surely, each face is surrounded by infinitely many open circuits.

(iv) What can we say about the occurence of an infinite cluster?

Definition 4.8 (disjoint occurrence).
Let A be an event, and $\omega \in \Omega$.

We say that $I \subseteq E$ <u>allows verification</u> of A for ω, if $\{\omega' \in \Omega : \omega'|_I = \omega|_I\} \subseteq A$. For example:

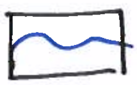
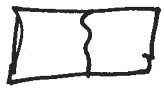
$A = \{$  $\}$, $\omega =$ , $I =$  OR .

We say that A and B <u>occur disjointly</u> for ω, if $\exists\, I_A, I_B \subseteq E$ disjoint such that

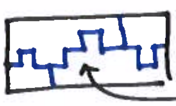
I_A allows verification of A for ω,
I_B " " " B " " .

Write

$$A \circ B := \{\omega \in \Omega : A \text{ and } B \text{ occur disjointly}\}.$$

For example: $A = \{$  $\}$, $B = \{$  $\}$

$\omega =$  $\in A \circ B$

$\omega' =$  $\notin A \circ B$ use same edges!