ANNOUNCEMENTS (SEE WEBSITE)

Theorem 4.9 (Van den Berg-Kesten inequality)

Let A, B denote <u>increasing</u> events, depending on finitely many edges. Then

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A) \cdot \mathbb{P}(B).$$

Proof. Note first that, as for the Harris inequality, the (geometric) graph structure does not play a role. Also, the edges are labelled $e_1, \dots, e_n$.

Consider $\hat{\mathbb{P}}_p := \mathbb{P}_p \times \mathbb{P}_p$. A sample $\hat{\omega}$ is written

$$\hat{\omega} = (\underbrace{\omega_1^r, \dots, \omega_n^r}_{\text{red}}, \underbrace{\omega_1^b, \dots, \omega_n^b}_{\text{blue}})$$

Define

$$\omega^k := (\omega_1^b, \dots, \omega_k^b, \omega_{k+1}^r, \dots, \omega_n^r).$$

Thus, $\omega^0 = \omega^r$ and $\omega^n = \omega^b$. Also, ω^k "interpolates".

Define
$$\hat{A}^k := \{\hat{\omega} : \hat{\omega}^k \in A\},$$
$$\hat{B} := \{\hat{\omega} : \omega^n = \omega^b \in B\}.$$

Note: $\hat{A}^0$ depends only on red edges, so

$$\hat{A}^0 \wedge \hat{B} = \hat{A}^0 \circ \hat{B} \quad \text{and} \quad \hat{\mathbb{P}}_p(\hat{A}^0 \circ \hat{B}) = \mathbb{P}_p(A)\,\mathbb{P}_p(B).$$

It suffices to show that

$$k \mapsto \hat{\mathbb{P}}_p\left(\hat{A}^k \circ B\right)$$

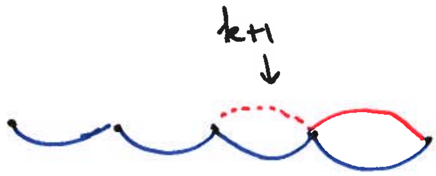
is ~~dec~~ non-increasing.

In fact, we prove, for each k, that

$$\hat{\mathbb{P}}_p\left(\hat{A}^{k+1} \circ B \,|\, \mathcal{G}\right) \le \hat{\mathbb{P}}_p\left(\hat{A}^{k} \circ B \,|\, \mathcal{G}\right), \quad \dagger$$

~~where $\mathcal{G} = \sigma$ (all edges except ω^b_{k+1} and ω~~

where $\mathcal{G} = \sigma\left(\text{all edges except } \omega^r_{k+1} \text{ and } \omega^b_{k+1}\right)$.



Conditional on $\mathcal{G}$, only ω^r_{k+1} and ω^b_{k+1} are random. Moreover, all events are increasing. Thus, we can brute-force and proceed by ~~casion~~ cases:

- for each of the two events, we see which ~~cod~~ combination of states ~~are~~ for ω^r_{k+1} ω^b_{k+1} makes the event occur,
- for each of the two events, we calculate the corresponding probability.

Case 1. Occurrence of $\hat{A}^k \circ \hat{B}$ (same for ω^r_{k+1})

if ω^b_{k+1} closed →
" " open →

b \ r	0	1
0	✓	✓
1	✓	✓

Occurrence of $\hat{A}^{k+1} \circ \hat{B}$ (only depends on ω^b_{k+1})

b \ r	0	1
0	✓	✓
1	✓	✓

In both cases, the probability is $\underline{1}$.

Case 2.

b \ r	0	1
0	X	✓
1	✓	✓

b \ r	0	1
0	X	X
1	✓	✓

$$\mathbb{P} = 1-(1-p)^2 \quad > \quad \mathbb{P} = p$$

In this case, one of the two events ~~eon~~ needs to use the $(k+1)$-th edge. This is easier if the edge is different for the two events, you have "two chances" to get it right!

Case 3.

b \ r	0	1
0	X	X
1	X	✓

b \ r	0	1
0	X	X
1	X	X

$$p^2 \quad > \quad 0$$

Both ~~edges~~ events require the $(k+1)$-th edge

Case 4a

b \ r	0	1
0	X	✓
1	X	✓

b \ r	0	1
0	X	X
1	✓	✓

$$p \quad = \quad p$$

Case 4b is symmetric to 4a. □

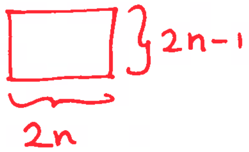
Exercise 4.10

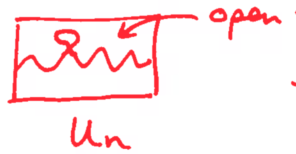
Let $\omega \in A$ for A increasing. Show that if a verifying set I is <u>minimal</u> for (A, ω), then

$$\omega|_I \equiv 1 .$$

Exercise 4.11.

Consider dimension $d = 2$ with $p = \frac{1}{2}$.

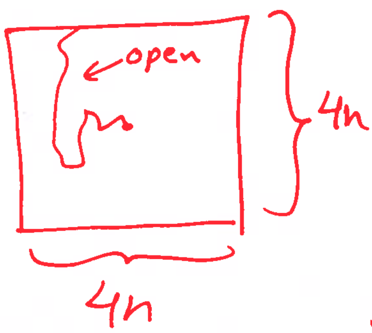
(i) Let $U_n :=$ 

, $A_n := \{$ 

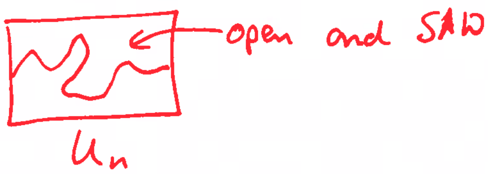
$\}$.

Show that $\mathbb{P}_{\frac{1}{2}}(A_n) = \frac{1}{2}$.

Deduce that

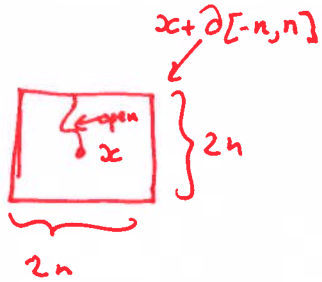
$\mathbb{P}_{\frac{1}{2}}\Big($ 

$\Big) \geq \frac{1}{2n}$.

(ii) Let $A_n' := \{$ 

$\}$.

What can we say about $\mathbb{P}_{\frac{1}{2}}(A_n')$?

(iii) Let $C_n(x) = \{$ 

$\}$.

Show that
$$\mathbb{P}_{\frac{1}{2}}\left(C_n(0) \circ C_n(0)\right) \geq \frac{1}{2n}.$$

What can we say about $\mathbb{P}_{\frac{1}{2}}\left(C_n(0)\right)$?

Russo's formula. **Definition 4.12.** (Pivotals)

Suppose that the edge set E is finite and that A is an increasing event. For $\omega \in \Omega$ and $e \in E$, write:

ω^e for ω except e is open,

ω_e " " " " " closed.

We say that e is <u>pivotal</u> for (A, ω) if
$$\omega^e \in A, \quad \omega_e \notin A.$$
Write $N(A,\omega) = N(A)$ for the # of pivotals.

Theorem 4.13 (Russo's formula). We have
$$\frac{d}{dp}\mathbb{P}_p(A) = \mathbb{E}_p[N(A)].$$

Proof. The proof is more or less immediate.

Let $(U_e)_{e\in E}$ denote an i.i.d. family of $U([0,1])$-random ~~variables~~ variables. For $\bar{p} = (p_e)_{e\in E}$, let

$$\omega_{\bar{p}} := (\mathbb{1}_{U_e \le p_e})_{e\in E}.$$

Let $e_1, \dots, e_n$ denote an enumeration of E. Then

$$(p_{e_1}, \dots, p_{e_n}) \mapsto \mathbb{P}(\omega_{\bar{p}} \in A)$$

is affine in each coordinate. If $p_j' > p_j$ and $p_i' = p_i$ $\forall i \neq j$, then

$$\mathbb{P}(\omega_{\bar{p}'} \in A) - \mathbb{P}(\omega_{\bar{p}} \in A) =$$

$$= (p_j' - p_j)\, \mathbb{P}(e_j \text{ is pivotal for } (A, \omega_{\bar{p}})).$$

Thus,

$$\frac{\partial}{\partial p_j} \mathbb{P}(\omega_{\bar{p}} \in A) = \mathbb{P}(e_j \text{ is pivotal for } (A, \omega_{\bar{p}})).$$

Summing over the edges gives the formula.

Remark. If A depends on finitely (n) many edges, then $p \mapsto P_p(A)$ is a polynomial (of degree n).

Chapter 5: The supercritical phase:

Recall:

- C_x is the connected component of x,
- $\{x \leftrightarrow \infty\} := \{|C_x| = \infty\}$,
- $\theta(p) := \mathbb{P}_p(x \leftrightarrow \infty)$,
- $\mathcal{E}_\infty = \bigcup_x \{x \leftrightarrow \infty\}$,
- $p_c = \sup\{p \in [0,1] : \theta(p) = 0\}$.

Notation. Define $\Lambda_n := [-n,n]^d \subseteq \mathbb{Z}^d \quad \forall n$.

Remark. $\forall p < p_c$, we have

$$\mathbb{P}_p(\mathcal{E}_\infty) \leq \sum_x \mathbb{P}_p(x \leftrightarrow \infty) = \sum_x 0 = 0.$$

Theorem 5.1. For any $p > p_c$, we have $\mathbb{P}_p(\mathcal{E}_\infty) = 1$. Thus, there is an infinite cluster $\mathbb{P}_p$-almost surely.

Definition 5.2. Let τ denote translation by $(1, 0, 0, \dots, 0) \in \mathbb{Z}^d$. For $\omega \in \Omega$, define $\tau(\omega)$ by $\tau(\omega)_e := \omega_{\tau(e)}$. For any event A, define

$$\tau^{-1}(A) := \{\omega \in \Omega : \tau(\omega) \in A\}.$$

If $\tau(A) = A$, then A is <u>translation-invariant</u>.