Theorem 5.3 (zero-one law). If A is shift-invariant, then $\mathbb{P}_p(A) \in \{0,1\}$.

This implies Theorem 5.2.

Proof. Let $\cdot \Delta \cdot$ denote symmetric difference. Note that $\mathbb{P}_p(\cdot \Delta \cdot)$ satisfies the triangular inequality. (It is a metric, except that $\mathbb{P}_p(B \Delta B')$ does not imply $B = B'$). Q

A *local sequence* is a sequence of events ~~sta~~ $(B_n)_n$ such that each B_n depends on finitely many edges.

Let $\mathcal{G}$ denote the set of events B such that $\mathbb{P}_p(B_n \Delta B) \to 0$ for some local sequence $(B_n)_{n \in \cdot}$

- $\mathcal{G}$ ~~dens~~ contains events which depend on finitely many edges.
- If $(A_n)_{n \geq 0} \subseteq \mathcal{G}$, then $\bigcup_n A_n \in \mathcal{G}$ by Q
- " " " " $\bigcap_n A_n \in \mathcal{G}$ by Q.
- If $A \in \mathcal{G}$ then $A^c \in \mathcal{G}$.

The monotone class argument says that $\mathcal{G} = \mathcal{F}$.

Proceed by contradiction. Suppose that

$$a := \mathbb{P}_p(A) \in (0,1).$$

Fix A' s.t. $\mathbb{P}_p(A \Delta A') \leq \varepsilon$ where $\varepsilon > 0$ is very small, and which only depends on edges in Λ_n for n large.

Let $A'' := \tau^{-10n}(A)$. Then:

$$\cancel{\mathbb{P}_p(A') \simeq \mathbb{P}_p(A)} \quad \text{~~(the diff. is smaller than ε)~~}$$

$$\cancel{\mathbb{P}_p(A} \quad \mathbb{P}_p(A' \Delta A) \leq \varepsilon$$

$$\mathbb{P}_p(A'' \Delta A) \leq \varepsilon$$

$$\mathbb{P}_p(A' \Delta A'') \leq 2\varepsilon \quad (\Delta\text{-inequality}).$$

But A' and A'' are independent and have probability approximately a; a contradiction. $\square$

Exercise 5.4. Show that the event

$$\{\exists \text{ an open bi-infinite SAW}\}$$

satisfies a zero-one law.