Definition 5.5. Let N_∞ denote the number of infinite clusters.

Theorem 5.6. Let $p \in [0,1]$. Then

- either $\mathbb{P}_p(N_\infty = 0) = 1$,
- or $\mathbb{P}_p(N_\infty = 1) = 1$.

Lemma 5.7. For any $p \in [0,1]$, we have

- $N_\infty = 0$ $\mathbb{P}_p$-a.s., or
- $N_\infty = 1$ $\mathbb{P}_p$-a.s., or
- $N_\infty = \infty$ $\mathbb{P}_p$-a.s..

Proof. By Theorem 5.3, it suffices to demonstrate that

$$\mathbb{P}_p(N_\infty = k) = 1 \quad \text{is false} \quad \forall k \in \{2, 3, \dots, \dots\}.$$

Fix k. Let $\mathcal{E}_n$ denote the event that all infinite connected components ~~tool~~ touch $\{-n, \dots, n\}^d =: \Lambda_n \subseteq \mathbb{Z}^d$.

Then
$$\{N_\infty = k\} \subseteq \cup_n \mathcal{E}_n.$$

Thus, $\mathbb{P}_p(\mathcal{E}_n \cap \{N_\infty = k\}) > 0$ for some n.

But $\mathcal{E}_n$ depends only on edges which are <u>not</u> contained in Λ_n. Thus, conditional on $\mathcal{E}_n$, all ~~edges are~~ those M edges are open with probability p^M.

Thus, $\mathbb{P}_p(\mathcal{E}_n \cap \{N_\infty = 1\}) \geq \mathbb{P}_p(\mathcal{E}_n) p^M > 0$,

which contradicts $\mathbb{P}_p(N_\infty = k) = 1$. $\square$

Theorem 5.6 now follows from the following lemma.

Lemma 5.8. For any $p \in [0,1]$, $\mathbb{P}_p(N_\infty \geq 3) \neq 1$.

Proof. Suppose $\mathbb{P}_p(N_\infty \geq 3) = 1$. A site $x \in \mathbb{Z}^d$ is a <u>trifurcation point</u> if:

- C_x, the connected component of x, is infinite,
- exactly three edges incident to x are open,
- $C_x \setminus \{x\}$ has three infinite components.

We now prove $\mathbb{P}_p(x \text{ is a trifurcation}) > 0$.

Let ~~$\omega \setminus \mathbb{E}$~~ $\mathbb{E}(\Lambda_n)$ denote the edges contained in Λ_n.

Let $\mathcal{E}_n$ denote that three ∞ connected components of $\omega \setminus \mathbb{E}(\Lambda_n)$ touch Λ_n. As before, $\mathbb{P}_p(\mathcal{E}_n) > 0$ for some n, and $\mathcal{E}_n$ depends only on edges outside $\mathbb{E}(\Lambda_n)$.

If $x \in \Lambda_n$, then $\mathbb{P}_p(x \text{ trifurcation}) \geq \mathbb{P}_p(\mathcal{E}_n)(p \wedge (1-p))^{|\mathbb{E}(\Lambda_n)|}$

> 0. $\oplus$.

Now, $\mathbb{P}_p(x \text{ trifurcation}) = \mathbb{P}_p(0 \text{ trifurcation}) =: c > 0$.

Thus, $\mathbb{E}_p[\# \text{ trifurcations in } \Lambda_n] = c(2n+1)^d$.

Claim: # trifurcations in Λ_n $\leq$ edges sticking out of Λ_n

$$= 2d \cdot (2n+1)^{d-1} = O(n^{d-1}).$$

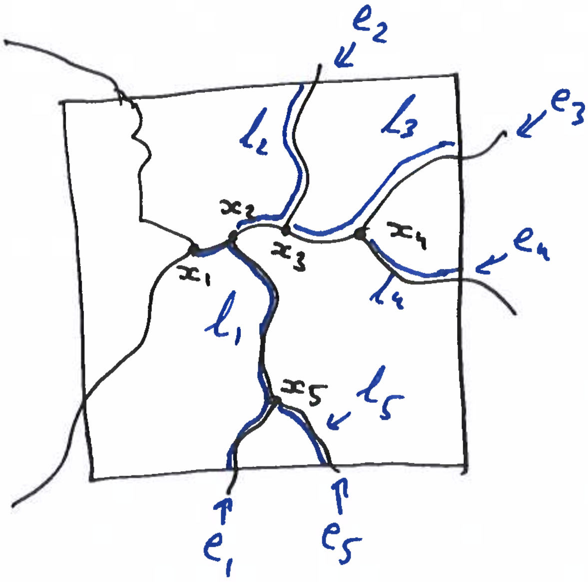
Proof: Let ω ~~a configurat~~ denote a configuration, and let $x_1, \dots, x_T$ be an enumeration of trifurcation points in Λ_n.

- From x_1, choose a SAW l_1 from x_1 to ∞ through ω,
- From x_2, " " " l_2 x_2 to ∞ through $\omega \setminus l_1$,
- From x_3, - - - - - l_3 from x_3 to ∞ through $\omega \setminus (l_1 \cup l_2)$

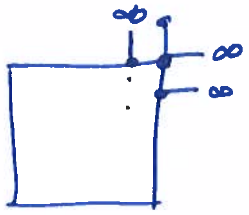
$\vdots$

This is possible, because each point is a trifurcation point! For each $1 \leq k \leq T$, let e_k denote the first edge of l_k which is not contained in Λ_n. Then

$$x_k \longmapsto e_k \quad \text{is injective.} \qquad \square.$$



Exercise 5.9. (A) is not really true.



How do we fix this?

Proposition 5.10. The function $[0,1] \to [0,1]$, $p \mapsto \theta(p)$ is:

(i) • right-continuous,

(ii) • left-continuous on $[0,1] \setminus \{p_c\}$.

Proof. First (i). Note:

$$p \mapsto \theta_n(p) := \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)$$

is an increasing polynomial, hence continuous. It is also decreasing in n, with limit $\theta(p)$. This proves $\theta(p)$ is right-continuous.

Now (ii). Let $\mathbb{P}$ denote the measure in which $(U_e)_{e\in E}$ is an i.i.d. family of $\mathcal{U}([0,1])$-distributed ~~RVs.~~ random variables. Let $(\omega_p)_{p\in[0,1]}$ denote the coupled percolation. Fix $p_0 > p_c$, we show

$$L := \lim_{p \uparrow p_0} \theta(p) = \theta(p_0) =: R$$

Note

$$R - L = \mathbb{P}\left(0 \in \mathcal{C}_{p_0} \;\&\; 0 \notin \mathcal{C}_p \;\forall p < p_0\right) =: D$$