Fix $p_1 = \frac{p_c + p_0}{2} \in (p_c, p_0)$.

Let x denote the smallest (w.r.t. some total ordering of $\mathbb{Z}^d$) vertex in $\mathcal{C}_{p_1}$. Then:

$$D :\overset{*}{=} \mathbb{P}\left(0 \overset{\omega_{p_0}}{\longleftrightarrow} x,\ 0 \overset{\omega_p}{\not\longleftrightarrow} x \ \ \forall p < p_0\right).$$

(Note: x is random). But

$$D \leq \mathbb{P}(\exists e,\ U_e = p) = 0. \qquad \square.$$

* Can restrict to p rational, so that uniqueness of the infinite cluster is still true a.s. $\forall p$.

The ~~super~~ sub critical phase; Chapter 6.

Theorem 6.1. Fix d. $\forall p < p_c(d)$, $\exists\ \psi = \psi(p) > 0$ such that, for n suff. large,

$$\mathbb{P}_p(|C| \geq n) \leq e^{-\psi n}.$$

Recall C is the component of 0.

Theorem 6.2. In the same context,*

$$\mathbb{P}_p(0 \longleftrightarrow \partial \Lambda_n) \leq e^{-\psi n}.$$

---

* $\psi(p)$ may be different

Proposition 6.3. If $\chi(p) := \mathbb{E}_p[|C|] < \infty$, then $\exists$ ~~φ~~ $\psi > 0$ such that

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \le e^{-\psi n} \quad \forall n \text{ large.}$$

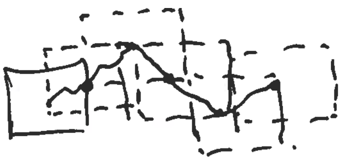
Proof. Since

$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(x \leftrightarrow 0) = \chi(p) < \infty,$$

$\exists\, k < \infty$ ~~s.t~~ such that

$$\underbrace{\sum_{x \in \partial\Lambda_k} \mathbb{P}_p(0 \overset{\text{in } \Lambda_k}{\longleftrightarrow} x)}_{=:\, a} < 1.$$

Now estimate $\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{mk})$ with the BK-$\le$.



$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{mk}) \le$$

disjoint occurrence

$$\sum_{l \ge m} \sum_{\substack{x_1, \dots, x_l \\ x_1 \in \partial\Lambda_k \\ x_{i+1} \in x_i + \partial\Lambda_k}} \mathbb{P}_p(x_1 \leftrightarrow x_2 \circ x_2 \leftrightarrow x_3 \circ \dots \circ x_{l-1} \leftrightarrow x_l)$$

$$\overset{BK}{\le} \sum_{l \ge m} \sum_{''} \mathbb{P}_p(x_1 \leftrightarrow x_2) \cdots \mathbb{P}(x_{l-1} \leftrightarrow x_l)$$

$$= \sum_{l \ge m} a^l = \frac{a^m}{1-a}$$

$\square$

Recall $\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$;

$$\chi(p) := \mathbb{E}_p[|C| \mathbb{1}_{|C|<\infty}].$$

Now, each site $x \in \mathbb{Z}^d$ is <u>in G</u> with probability $\gamma \in [0,1]$, independently of anything else. Write $\mathbb{P}_{p,\gamma}$ for the product measure. Define

$$\theta(p,\gamma) := \mathbb{P}_{p,\gamma}(0 \leftrightarrow G)$$

$$\chi(p,\gamma) := \mathbb{E}[|C| \mathbb{1}_{C \cap G = \emptyset}].$$

Comment. We proved Prop. 5.10 by observing that, "around" $p_0 > p_c$, connecting to ∞ equals connecting to $\mathcal{C}_{p_1}$; $p_c < p_1 < p_0$. These definitions follow the same spirit: now we connect to G which is "at finite distance".

Lemma 6.4.

(i) $\lim_{\gamma \downarrow 0} \theta(p,\gamma) = \theta(p)$;

(ii) $\lim_{\gamma \downarrow 0} \chi(p,\gamma) = \chi(p)$;

(iii) $$\frac{\partial \theta(p,\gamma)}{\partial \gamma} = \frac{\chi(p,\gamma)}{1-\gamma}.$$

Proof. For (i) & (ii), use

$$\theta(p) := 1 - \sum_{k=1}^{\infty} \mathbb{P}_p(|C| = k); \quad \chi(p) := \sum_{k=1}^{\infty} k \mathbb{P}_p(|C| = k).$$

Thus $$\theta(p,\gamma) = 1 - \sum_{k=1}^{\infty} (1-\gamma)^k \mathbb{P}_p(|C|=k);$$
$$\chi(p,\gamma) = \sum_{k=1}^{\infty} (1-\gamma)^k k \mathbb{P}_p(|C|=k).$$

Differentiating $\theta(p,\gamma)$ in γ gives

$$\frac{\partial \theta(p,\gamma)}{\partial \gamma} = \sum_{k=1}^{\infty} k(1-\gamma)^{k-1} \mathbb{P}_p(|C|=k)$$
$$= \frac{\chi(p,\gamma)}{1-\gamma}. \qquad \square.$$

Lemma 6.5 We have

(i) $$(1-p)\frac{\partial\theta}{\partial p} \le 2d(1-\gamma)\theta\frac{\partial\theta}{\partial\gamma},$$

(ii) $$\theta \le \gamma\frac{\partial\theta}{\partial\gamma} + \theta^2 + p\theta\frac{\partial\theta}{\partial p}.$$

Statement and proof not examinable!

We shall use Reimers inequality. This is an improved version of BK for events which ~~are~~ occur disjointly, but which are not disjoint.

Proof. (i). Russo:
$$\frac{\partial\theta}{\partial p} = \sum_{e\in E^d} \mathbb{P}_{p,\gamma}(e \text{ is pivotal for } 0 \leftrightarrow G)$$
$$= \sum_{e\in E^d} \frac{\mathbb{P}_{p,\gamma}(e \text{ is closed \& pivotal for } 0\leftrightarrow G)}{1-p}$$

$$= \sum_{xy \in \vec{E}^d} \frac{1}{1-p} \mathbb{P}_{p,\gamma}\left(\{0 \leftrightarrow x \ \& \ 0 \not\leftrightarrow G\} \circ \{y \leftrightarrow G\}\right)$$

Reimers

$$\le \frac{1}{1-p} \sum_{xy} \underbrace{\mathbb{P}_{p,\gamma}(0 \leftrightarrow x \ \& \ 0 \not\leftrightarrow G)}_{2d\,\underbrace{\chi(p,\gamma)}_{=(1-\gamma)\frac{\partial\theta}{\partial\gamma}}} \underbrace{\mathbb{P}_{p,\gamma}(y \leftrightarrow G)}_{\theta(p,\gamma)}$$

$$= \frac{1}{1-p}\, 2d\, \theta \ (1-\gamma)\frac{\partial \theta}{\partial \gamma} .$$

(ii) $\{0 \leftrightarrow G\} =$

$$\{|C \cap G| = 1\} \qquad \text{(A)}$$

$$\cup \left(\{|C \cap G| \ge 2\} \cap (\{0 \leftrightarrow G\} \circ \{0 \leftrightarrow G\})\right) \qquad \text{(B)}$$

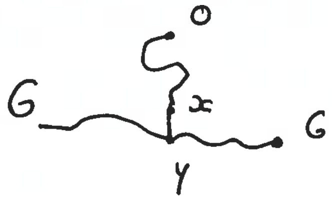
$$\cup \left(\{|C \cap G| \ge 2\} \setminus (\{0 \leftrightarrow G\} \circ \{0 \leftrightarrow G\})\right) \qquad \text{(C)} .$$

We bound $\mathbb{P}_{p,\gamma}(\text{(A)})$, $\mathbb{P}_{p,\gamma}(\text{(B)})$, and $\mathbb{P}_{p,\gamma}(\text{(C)})$ individually.

$$\mathbb{P}(\text{(A)}) = \sum_{k=1}^{\infty} k\, \gamma (1-\gamma)^{k-1}\, \mathbb{P}_p(|C| = k) = \gamma \frac{\partial \theta}{\partial \gamma} .$$

$$\mathbb{P}(\text{(B)}) \overset{BK}{\le} \mathbb{P}(0 \leftrightarrow G)^2 = \theta^2 .$$

Now (C)



$\exists\, xy \in \vec{E}^d$ s.t.:

- xy is open,
- ~~xy~~ $0 \overset{\omega\setminus xy}{\longleftrightarrow} x$, $0 \overset{\omega\setminus xy}{\not\longleftrightarrow} G$
- $\{y \overset{\omega\setminus xy}{\longleftrightarrow} G\} \circ \{y \overset{\omega\setminus xy}{\longleftrightarrow} G\}$.

$$\mathbb{P}_{p,\gamma}(\text{(C)}) \le \sum_{xy} \mathbb{P}_{p,\gamma}\left(\{xy \text{ open}, 0 \overset{\omega\setminus xy}{\longleftrightarrow} x, 0 \overset{\omega\setminus xy}{\not\longleftrightarrow} G\} \circ \{y \leftrightarrow G\} \circ \{y \leftrightarrow G\}\right)$$

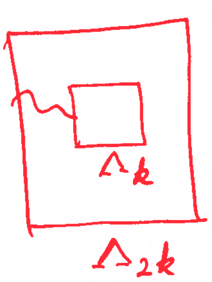
$$\overset{\text{Reimers}}{\le} \underbrace{\sum_{xy} \mathbb{P}_{p,\gamma}\left(\{xy \text{ open}, 0 \overset{\omega\setminus xy}{\longleftrightarrow} x, 0 \overset{\omega\setminus xy}{\not\longleftrightarrow} G\} \circ \{y \leftrightarrow G\}\right)}_{\underbrace{p \sum_{xy} \mathbb{P}_{p,\gamma}(xy \text{ pivot for } 0 \leftrightarrow G)}_{\frac{\partial \theta}{\partial p}}} \underbrace{\mathbb{P}_{p,\gamma}(y \leftrightarrow G)}_{\theta}$$

$$= p\,\theta\,\frac{\partial \theta}{\partial p}. \qquad \square$$

Theorem 6.2 can be proved from the lemma, ~~But, the proof is not~~ using the differential inequalities. This is not hard, but also not super insightful for the ~~zero~~ remainder of the course. We omit the proof.

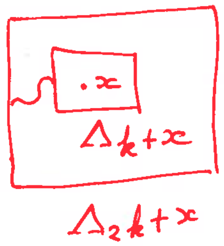
Exercise 6.6. ~~Proof~~ Prove Thm 6.2 $\Rightarrow$ Thm 6.1.

Fix $d, p < p_c$.

(i) Prove Thm 6.2 $\Rightarrow$ $a_k := \mathbb{P}_p\Big($  $\Big) \le e^{-\alpha k}$,

for k large and for $\alpha = \alpha(p)$ fixed.

(ii) Fix k so large that $a_k \le \left(2^{-10^8}\right)^d$.

Let $A(x) := \Big\{$  $\Big\}$.

We shall only consider $(A(x))_{x \in (k\mathbb{Z})^d}$

important.

Prove: if $X \subseteq k\mathbb{Z}^d$ s.t. $\forall x, y \in X$,

$x = y$ or ~~$\Lambda_{2k}+x$~~

$\Lambda_{2k}+x$ & $\Lambda_{2k}+y$ are disjoint,

then

$$\mathbb{P}_p\Big(\bigcap_{x\in X} A(x)\Big) = (a_k)^{|X|}.$$

(iii) Prove: if $|C|$ is "very large", then we can find a set X of the above type and of the same order as C, s.t.

$$\bigcap_{x \in X} A(x) \text{ occurs.}$$

(iv) Moreover, X can be chosen s.t. $\forall x, y \in X$, $\exists$ a path $p \subseteq X$ from x to y s.t.

the distance from p_k to p_{k+1} is at most $10 \cdot d \cdot k$.

(v) By studying lattice animals, conclude Theorem 6.1