the distance from p_k to p_{k+1} is at most $10 \cdot d \cdot k$.

(v) By studying lattice animals, conclude Theorem 6.1

Definition 6.7. ~~For p~~ Fix any dimension $d \geq 1$. Let $e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{Z}^d$. For any $p < p_c(\mathbb{Z}^d)$, define

$$\frac{1}{L(p)} := -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_p[0 \leftrightarrow ne_1].$$

Lemma 6.8. This is well-defined and smaller than ∞.

Proof. ~~BK~~ Harris $- \leq$: $\log \mathbb{P}_p[0 \leftrightarrow ne_1] + \log \mathbb{P}_p[0 \leftrightarrow me_1]$

$$\leq \log \mathbb{P}_p[0 \leftrightarrow (n+m)e_1].$$

This implies convergence. Sharpness implies

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow ne_1) < 0.$$

Note also that $\mathbb{P}_p(0 \leftrightarrow ne_1) \geq p^n$ so that

$$\frac{1}{L(p)} \geq -\log p.$$

$\cancel{\square}$.

Thus $L(p) \in [\frac{1}{\log p}, \infty)$. $\square$.

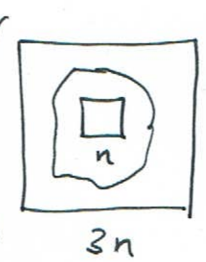
Chapter 7. Russo-Seymour-Welsh theory.

We restrict to dimension 2.

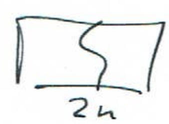
Theorem 7.1. $\exists\, \psi : [0,1] \to [0,1]$ which is an increasing homeomorphism s.t., $\forall p \in [0,1]$, $\forall n \geq 1$,

$$\mathbb{P}_p^{\mathbb{Z}^2}\Big($$  $$n\Big) \geq \psi\Big(\mathbb{P}_p^{\mathbb{Z}^2}\Big($$  $$n\Big)\Big).$$

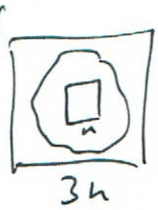
Corollary 7.2. $\exists\, c > 0$ s.t. $\forall n$,

$$\mathbb{P}_{\frac{1}{2}}^{\mathbb{Z}^2}\Big($$  $$\Big) \geq c.$$

Proof.

$$\mathbb{P}_{\frac{1}{2}}\Big($$  $$n\Big) \geq \psi\Big(\mathbb{P}_{\frac{1}{2}}\Big($$  $$n\Big)\Big) \geq \psi\left(\tfrac{1}{2}\right).$$

Harris ~~BK~~ $-\leq$:

$$\mathbb{P}_{\frac{1}{2}}\Big($$  $$\Big) \geq \mathbb{P}_{\frac{1}{2}}\Big($$  $$n\Big)^{12} \geq \psi\left(\tfrac{1}{2}\right)^{12} > 0.$$

$\square$

Corollary 7.3. (i) $p_c(\mathbb{Z}^2) = \frac{1}{2}$,

(ii) $\theta(p_c) = 0$.

Proof. Apply Corollary 7.2 to the dual to see that $\theta(\frac{1}{2}) = 0$. Thus, $p_c \geq \frac{1}{2}$.

If $p_c > \frac{1}{2}$, then

$\mathbb{P}_{\frac{1}{2}}\Big($  $n\Big)$ decays exponentially in n due to sharpness. $\square$

Remarks. (i) Thm 7.1 : see Köhler-Schindler, Tassion; very nice paper which applies beyond Bernoulli percolation.

(ii) Thm 7.1. Is called a Russo-Seymour-Welsh theorem.

(iii) There are many different RSW theorems for different (non-Bernoulli) models.

(iv) We do not prove Thm 7.1, but rather the following lemma.

Lemma 7.4. $\exists\, c > 0$ s.t.

$$\mathbb{P}_{\frac{1}{2}}\Big($$


$$\Big) \geq c.$$

This lemma clearly also implies Corollary 7.2 and Corollary 7.3. This is our objective.

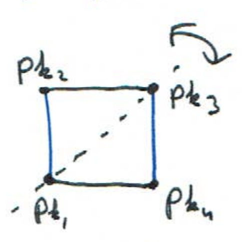
Definition 7.5. A <u>symmetric quad</u> is a closed circuit through $\mathbb{Z}^2$ $(p_k)_{0 \le k \le n}$ with four marked points $0 \le k_1 < k_2 < k_3 < k_4 < n$, and a symmetry $\varphi \in \mathrm{Aut}(\mathbb{Z}^2)$ s.t.:

(i) $\varphi(p) = p$

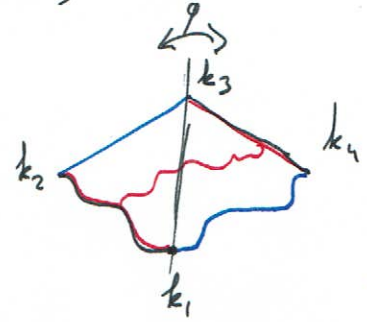
(ii) $\varphi(\{p_{k_1}, p_{k_2}, p_{k_3}, p_{k_4}\}) = \{p_{k_1}, \dots, p_{k_4}\}$.

(iii) $\varphi(p_{k_1}) = p_{k_1}$; $\varphi(p_{k_3}) = p_{k_3}$,

(iv) $\varphi(p_{k_2}) = p_{k_4}$



Lemma 7.6. If (p, k_1, k_2, k_3, k_4) is a symmetric quad, then

$$\mathbb{P}_{\frac{1}{2}}\Big($$  $$\Big) \geq \frac{1}{2}.$$

Proof. Exercise. Hint: compare ω with $\omega^* + (\frac{1}{2}, \frac{1}{2})$, that is, the dual-complement of ω shifted by $(\frac{1}{2}, \frac{1}{2})$.

To prove lemma 7.4, we use three ingredients:

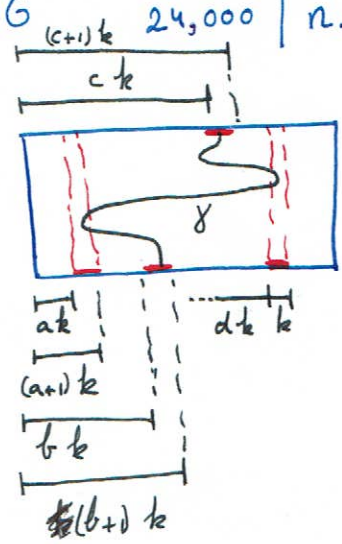
(i) ~~BK~~ Harris $- \leq$;

(ii) The square-root trick; $\max_k \mathbb{P}_p(A_k) \geq 1 - \sqrt[n]{1 - \mathbb{P}_p(A_1 \cup \dots \cup A_n)}$;

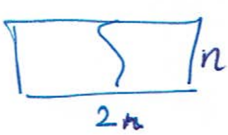
(iii) Lemma 7.6.

Proof of Lemma 7.4. Fix n. WLOG $24{,}000 \mid n$. Let $k = \frac{n}{100}$.

Let

$$\Gamma_n(a, b, c, d) := \Big\{ \exists \gamma \text{ open} :$$  $$\Big\}.$$

Note: (i) $\Gamma_n(a,b,c,d)$ is increasing,

(ii) $\bigcup_{0 \leq a,b,c,d \leq 199} \Gamma_n(a,b,c,d) = \Big\{$  $\Big\}$.

~~Square-root trick:~~

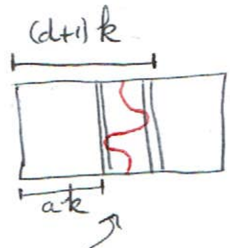
~~$\max_{a,b,c,d} \mathbb{P}_p(\Gamma(a,b,c,d))$~~

By the square-root trick,

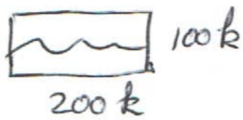
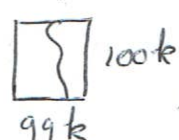
$$\max_{a,b,c,d} \mathbb{P}_{\frac{1}{2}}\left(\Gamma_n(a,b,c,d)\right) \geq 1 - \sqrt[200^4]{\tfrac{1}{2}} =: c_1 > 0.$$

Most of the proof of Lemma 7.4 is easy, but we must distinguish between quite some cases. Each case yields a different constant. At the end, we take "c" in Lemma 7.4 to be the minimum of these constants.

CASE I. $d - a < 99$.

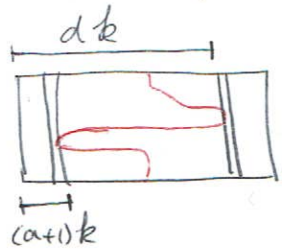


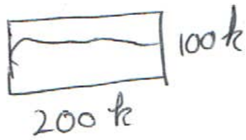
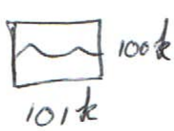
Then

$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \overset{\text{Harris}}{\geq} \mathbb{P}_{\frac{1}{2}}\Big($  $\Big)^{1000}$

$$\geq c_1^{1000} =: c_{\mathrm{I}}.$$

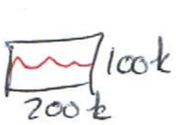
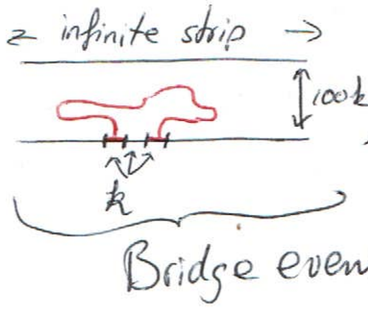
CASE II. $d - a > 101$.



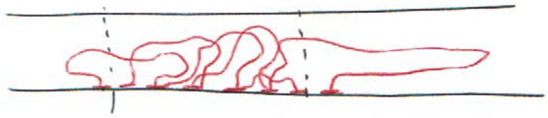
$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \overset{\text{Harris}}{\geq} \mathbb{P}_{\frac{1}{2}}\Big($  $\Big)^{1000} \geq c_1^{1000} =: c_{\mathrm{II}} = c_{\mathrm{I}}.$

Remaining cases: $d - a \approx 100$ (that is: $d - a \in \{99, 100, 101\}$).

INTERMEZZO.

$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \geq \mathbb{P}_{\frac{1}{2}}\Big($  $\Big)^{200}$, (see next page)

Since

 $\supseteq$  .

Our goal for the remaing case is to show that the bridge event has a good $\mathbb{P}_{\frac{1}{2}}$-probability.

WLOG, $a = 0$, $d \in \{99, 100, 101\}$.

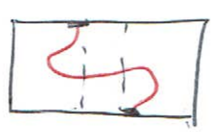
~~CASE III $b \notin [40, 60]$ or $c \notin [40, 60]$.~~

CASE III. $|c - b| > 1$.

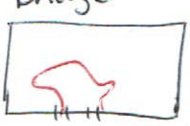
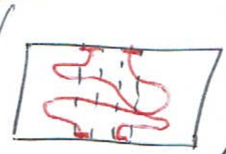
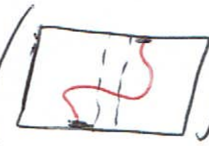
Suppose WLOG $c \geq b + 2$, that is,

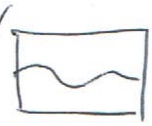
$\mathbb{P}_{\frac{1}{2}}\Bigg($  $\Bigg) \geq c_1.$

Then by symmetry,

$\mathbb{P}_{\frac{1}{2}}\Bigg($  $\Bigg) \geq c_1.$

Harris-$\geq$;

$\mathbb{P}_{\frac{1}{2}}\Bigg($  $\Bigg) \geq \mathbb{P}_{\frac{1}{2}}\Bigg($  $\Bigg) \geq \mathbb{P}_{\frac{1}{2}}\Bigg($  $\Bigg)^2 \geq c_1^2,$

and $\mathbb{P}_{\frac{1}{2}}\Bigg($  $\Bigg) \overset{\text{Intermezzo}}{\geq} c_1^{400} =: C_{\text{III}}$.

The remaining case is:

- $a = 0$,
- $d \in \{99, 100, 101\}$, and
- $c - b \in \{-1, 0, 1\}$.

Also, assume WLOG that $c \leq 60$. [The case $c \geq 40$ is identical.)

This is Case IV. In this case

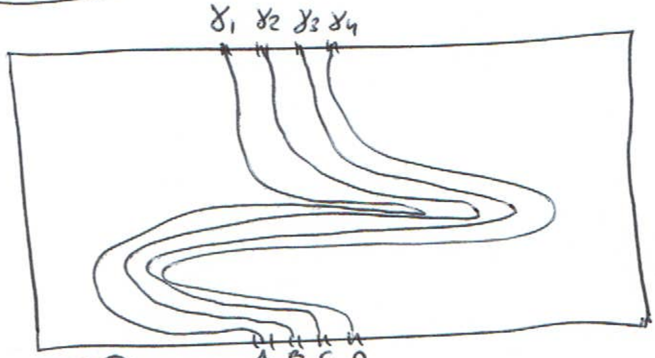
$\Gamma_n(a, b, c, d)$

$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \geq c_1 .$

ck

Let $\Gamma_n(a, b, c, d) + (mk, 0)$ denote the shift of this event by $(mk, 0)$. Note that

$$\underbrace{\mathbb{P}\Big(\bigcap_{m \in \{0,2,4,6\}} \Gamma_n(a,b,c,d) + (mk, 0)\Big)} \overset{\text{Harris}}{\geq} c_1^4 .$$

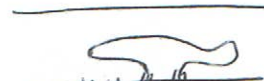


$=: M$

~~There are 3 options:~~

This event is ~~the~~ contained in the union of three events:

(i) 

(ii) 

(iii) $M \setminus (\{\text{Bridge at } A, B\} \cup \{\text{Bridge at } C, D\})$.

Note: the ~~maxi~~ event with largest likelihood has probability at least $\frac{c_1^4}{3}$ by a union bound.

Case IV.i. $\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \geq \frac{c_1^4}{3}$,

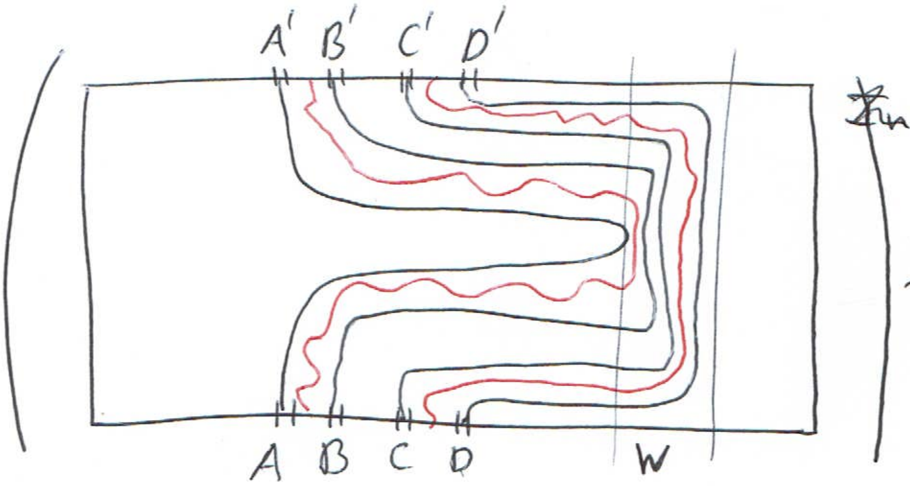
so $\mathbb{P}_{\frac{1}{2}}\Big($ 

$\Big) \geq \left(\frac{c_1^4}{3}\right)^{200} =: c_{IV.i}$

Case IV.ii. idem with $c_{IV.ii} = c_{IV.i}$.

We are left with the hardest case, namely Case IV.iii; ~~without~~ and we use Lemma 7.6.

Case IV.iii. We know

$\mathbb{P}_{\frac{1}{2}}\Big($ 

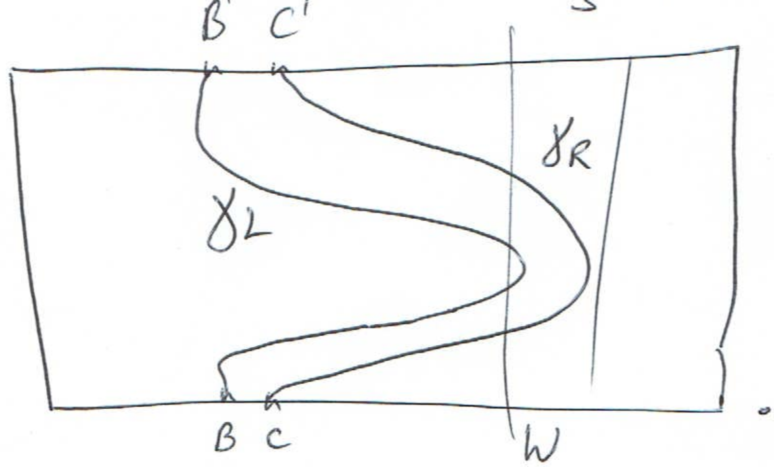
$\Big) \geq \frac{c_1^4}{3}$

In this picture, red paths are open paths for the dual-complement

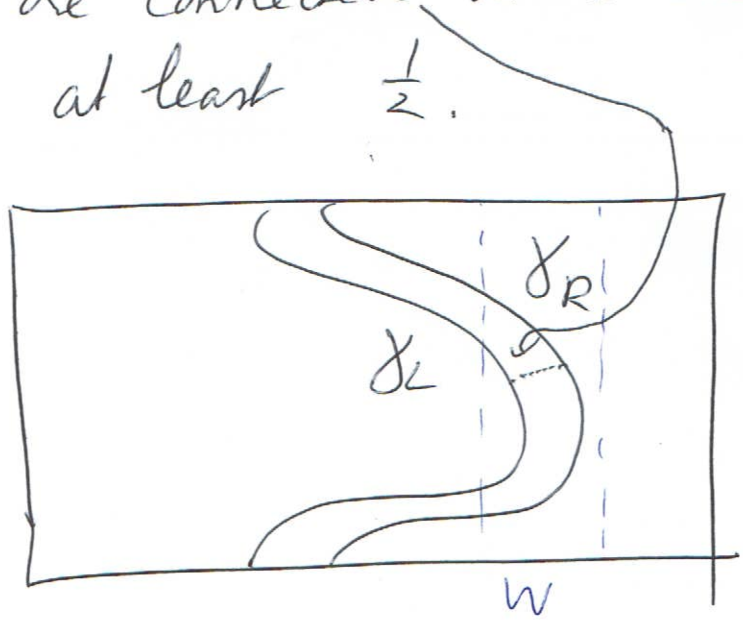
The ~~presents~~ presence of the red paths proves that A is not connected to B and that C is not connected to D.

Now let γ_L be the <u>leftmost</u> crossing from B to B', and γ_R the <u>rightmost</u> crossing from C to C'.

Since we are in Case IV. iii, the probability that the <u>rightmost</u> point of γ_L and γ_R lie in W, is at least $\frac{c_1^4}{3}$.

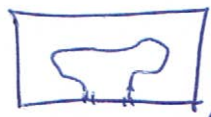


Claim: <u>Conditional</u> on the existence and the above property for γ_L & γ_R, the probability that they are connected within the rectangle, is at least $\frac{1}{2}$.



The claim implies

$\mathbb{P}_{\frac{1}{2}}\Big($ 

$\Big) \geq \mathbb{P}_{\frac{1}{2}}\Big($  $\Big)^{200} \geq$

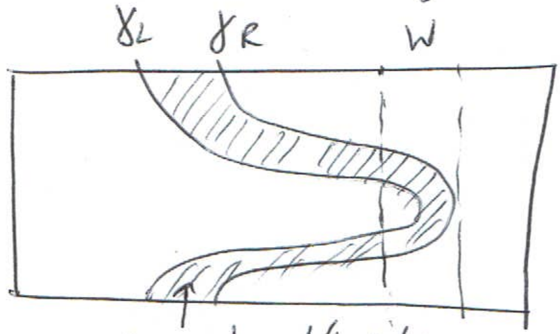
$$\geq \left(\frac{C_1^4}{3}\cdot\frac{1}{2}\right)^{200} =: C_{IV.iii},$$

and therefore the lemma with

$$C = \min\{C_{I}, C_{II}, C_{III}, C_{IV.i}, C_{IV.ii}, C_{IV.iii}\} > 0.$$

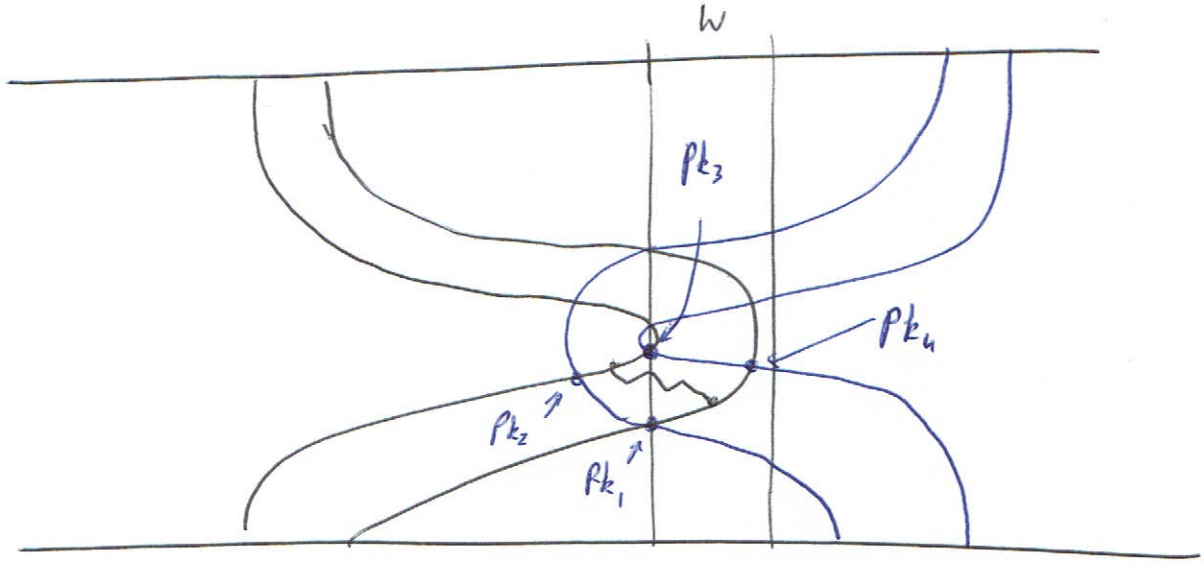
Proof of the claim.

Condition on the precise location of γ_L and γ_R. Conditional on this, the edges <u>between</u> γ_L and γ_R are still Bernoulli$(\frac{1}{2})$, because γ_L is the leftmost and γ_R the rightmost path.



Indeed, $\{\gamma_L = P, \gamma_R = P'\}$ is measurable w.r.t. the edges <u>not</u> in between γ_L and γ_R. We want to find a symmetric domain in (γ_L, γ_R).

For this, reflect the paths around the vertical line which bounds D on the left.



Then we find a symmetric domain for some appropriate choice. By the lemma, the probability that γ_L and γ_R are connected in the rectangle, is at least $\frac{1}{2}$. □