PERCOLATION
Exercises

Exercise 1. Let T be a Bienaymé-Galton-Watson tree with offspring distribution Poisson(λ), for some $\lambda > 1$. Let r be its root.

1. Express the probability $\theta(\lambda)$ that T is **infinite**, by giving an implicit equation satisfied by $\theta(\lambda)$. The results of the course can be used without a proof.

Let $p \in (0,1)$, we consider Bernoulli bond percolation with parameter p on T. We denote by $\theta(\lambda, p)$ the probability that the cluster of r for this percolation configuration on T is **infinite**.

2. For which values of p do we have $\theta(\lambda, p) = 0$? We define p_c as the supremum of these values; compute p_c.

3. Give an implicit expression for $\theta(\lambda, p)$ when $p > p_c$.

4. Show that as $p \to p_c^+$, we have

$$\theta(\lambda, p) = c_\lambda (p - p_c) + o(p - p_c).$$

   and compute the constant c_λ.

Exercise 2. Let $G = (V, E)$ be a (simple, infinite, countable, locally finite, connected) graph. Let $p \in (0,1)$. We consider **site** percolation on G: let $\xi = (\xi_v)_{v \in V}$ be i.i.d Bernoulli(p) random variables; we denote by $\mathbb{P}_p^{\text{site}}$ this probability measure on $\{0,1\}^V$.

For $v_0 \in V$, the notation $v_0 \leftrightarrow \infty$ stands for the event that there exists an infinite path $(v_0, v_1, \dots)$ of pairwise different vertices such that

$$\forall i \in \mathbb{N}, \xi(v_i) = 1 \text{ and } \{v_i v_{i+1}\} \in E.$$

1. Quickly recall an argument of the course to show that

$$\theta_v^{\text{site}}(p) := \mathbb{P}_p^{\text{site}}(v \leftrightarrow \infty)$$

   is increasing in p.

We define $p_c^{\text{site}} := \inf\{p \in (0,1) \mid \theta_v^{\text{site}}(p) > 0\}$.

2. Recall why this definition of p_c^{site} is independent of $v \in V$.

We also consider the usual bond percolation on G, and denote by $\mathbb{P}_p^{\text{bond}}, \theta_v^{\text{bond}}(p), p_c^{\text{bond}}$ the objects from the course. Our goal is to compare p_c^{site} with p_c^{bond}.

We consider the following algorithm. Suppose that $\xi \in \{0,1\}^V$ is given. Then the following will construct a configuration $\omega \in \{0,1\}^E$. We fix an order $V = \{v_1, v_2, \dots\}$.

(i) Let V_0, W_0 be two subsets of V, initially empty.

(ii) if $\xi(v_1) = 0$, go to step (iv). Otherwise, let $V_1 = \{v_1\}, W_1 = \emptyset$.

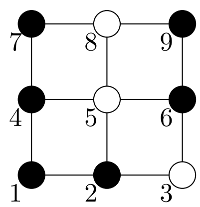
(iii) suppose that V_k, W_k are constructed, then let n be the smallest index of a vertex v_n in $V \setminus (V_k \cup W_k)$ that has a neighbour in V_k. If it does not exist, go to step (iv). Otherwise, let v'_m be its neighbour of smallest index in V_k, and let $e = \{v_n v'_m\} \in E$.

- if $\xi(v_n) = 1$, set $\omega(e) = 1$ and add v_n to V_k.
- if $\xi(v_n) = 0$, set $\omega(e) = 0$ and add v_n to W_k.

Let V_{k+1}, W_{k+1} be these new subsets of V.

(iv) Complete ω on all edges where it is not defined, with independent Bernoulli(p) random variables.

3. Run the algorithm (up to the last occurrence of step (iii)) on the following example, where the order of vertices is given, and a black (resp. white) vertex means that $\xi(v) = 1$ (resp. 0):



4. Using this algorithm, show that for any $v \in V$ and $p \in (0,1)$,

$$\theta_v^{\text{site}}(p) \le \theta_v^{\text{bond}}(p).$$

5. Deduce that $p_c^{\text{site}} \ge p_c^{\text{bond}}$.

6. (*) Using similar ideas, show that if the maximal degree d in G is finite, we have $p_c^{\text{site}} \le 1 - (1 - p_c^{\text{bond}})^{d-1}$.

Exercise 3. We consider bond percolation on $\mathbb{Z}^d$. Show that for any S finite containing 0 and any $A \cap S = \emptyset$,

$$\mathbb{P}_p[0 \longleftrightarrow A] \le p \sum_{\{x,y\} \in E \text{ s.t. } x \in S \text{ and } y \notin S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x] \mathbb{P}_p[y \longleftrightarrow A]$$

in two different ways (first using the BK inequality, then by conditioning on the cluster of 0 in S).

Exercise 4. We consider bond percolation on $\mathbb{Z}^2$. Let $A_{n,k}$ be the event that there exists k distinct paths that cross the square $[0,n]^2$ vertically. Show that

$$\mathbb{P}_p[A_{n,k}] \le \mathbb{P}_p[A_{n,1}]^k$$

in two different ways (first using the BK inequality, then by induction and conditioning on the subset of vertices that are connected by open edges to the left boundary of $[0,n]^2$).

Exercise 5. Let n be odd, and let $G = (V,E)$ be a graph with n edges. Let B_n be the event that there are more open edges than closed edges.

1. Describe the event that $e \in E$ is pivotal.
2. Show that

$$\mathbb{P}_p[\mathrm{Piv}_e(A_n)] = \binom{2n}{n} p^n (1-p)^n.$$

3. Show that $n\binom{2n}{n} \sim 2^{2n-1}/\sqrt{\pi n}$ as $n \to \infty$. Deduce that

$$\frac{d}{dp}\mathbb{P}_p[A_n] \geq c\sqrt{n}[p(1-p)/4]^n.$$

   Give the aspect of the function $p \mapsto \mathbb{P}_p[A_n]$ for higher and higher values of n.