PERCOLATION
Exam 2
3 exercises; 4 pages

Exercise 1 (Strict convexity)**.** Throughout this exercise, we consider bond percolation on $\mathbb{Z}^d$ with $d \in \mathbb{Z}_{\geq 2}$ and fixed percolation parameter $p \in (0, p_c)$.

1. Define rigorously the *correlation length* $L(p)$, and show that it is well-defined and lies in $(0, \infty)$. You may use the following results from the course provided that they are clearly stated: the Harris inequality and exponential decay of the cluster size in the subcritical regime.

2. Prove that the map
$$m_p : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty),\ (a, b) \mapsto -\log \mathbb{P}_p(a \longleftrightarrow b)$$
is a metric which is $(-\log p)$-Lipschitz in each coordinate.

The remainder of this exercise is hard. For any $a \in \mathbb{R}^d$, define $\lfloor a \rfloor := (\lfloor a_1 \rfloor, \ldots, \lfloor a_d \rfloor) \in \mathbb{Z}^d$. By arguing as in the first part of this exercise and using the Lipschitz property, it is not hard to see that
$$\| \cdot \|_p : \mathbb{R}^d \to [0, \infty),\ a \mapsto \lim_{n\to\infty} \frac{1}{n} m_p(0, \lfloor na \rfloor)$$
defines a norm on $\mathbb{R}^d$. You do not need to prove this. The unit ball of $\|\cdot\|_p$ is expected to be strictly convex, but this is actually an (important) open problem. In the remainder of this exercise, you should **assume this conjecture**. For any $x^1, \ldots, x^k \in \mathbb{R}^d$, let $A_n(x^1, \ldots, x^k)$ denote the event that $\lfloor nx^1 \rfloor$, ..., $\lfloor nx^k \rfloor$ belong to the same connected component.

3. Fix $x, y, z \in \mathbb{Z}^d$. Prove that
$$\mathbb{P}_p\Big(A_n(x, y, z) \Big| A_n(x, y)\Big)$$
decays exponentially fast in n if and only if z is *not* contained in the line segment from x to y. In other words, if and only if $z \notin [x, y] := \{x + \lambda(y - x) : \lambda \in [0, 1]\}$. Note: the *if and only if* has two directions both of which are awarded points. You may use any result from the course provided it is clearly stated.

4. Let $B_\varepsilon \subset \mathbb{R}^d$ denote the Euclidean ball of radius $\varepsilon > 0$ centred at the origin. Describe, using a simple argument (there is no need to go into technical details), how the previous part of the exercise implies that
$$\mathbb{P}_p\Big(C_{\lfloor nx \rfloor} \subset n([x, y] + B_\varepsilon) \Big| A_n(x, y)\Big)$$
tends to one exponentially fast in n for any fixed $\varepsilon > 0$. Here C_a denotes the connected component of a.

Thus, the last part proves that the conditional shape of $C_{\lfloor nx \rfloor}/n$ looks like an almost straight line segment from x to y.

Exercise 2 (Bond percolation on the triangular and hexagonal lattices)**.** The aim of this problem is to compute the critical probability for bond percolation on the triangular lattice $\mathbb{T}$ and on the hexagonal lattice $\mathbb{H}$. We denote by $\mathbb{P}_p^{\mathbb{T}}, \theta_{\mathbb{T}}(p)$ and $p_c(\mathbb{T})$ (resp. $\mathbb{P}_p^{\mathbb{H}}, \theta_{\mathbb{H}}(p)$ and $p_c(\mathbb{H})$) the usual objects on these graphs. On both these graphs you can use the following facts without a proof:

- Let C be the cluster of 0. If $p < p_c$, then there exists a $c > 0$ such that $\mathbb{P}_p(|C| > n) \leq \exp(-cn)$.
- If $p > p_c$, then a.s. there exists a unique infinite connected component.
- The Harris inequality, square-root trick, and BK inequality.

We label the vertices of $\mathbb{T}$ by elements of $\mathbb{Z}^2$ as in Figure 1, and we consider the region $\Lambda_n \subset \mathbb{T}$ with vertices labeled by $[0, n-1]^2$ and all bonds between them. We also consider its dual $\Lambda_n^* \subset \mathbb{H}$ as in Figure 1. We denote by $\mathcal{A}_n$ the event that Λ_n has a left-right crossing.

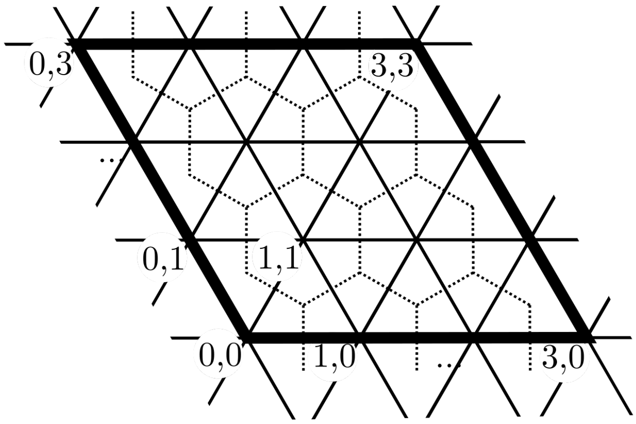


Figure 1: The subset $\Lambda_4 \subset \mathbb{T}$ and its dual $\Lambda_4^* \subset \mathbb{H}$ (dotted)

1. Show that if $p < p_c(\mathbb{T})$, then $\mathbb{P}_p^{\mathbb{T}}(\mathcal{A}_n) \to 0$ as $n \to \infty$. Deduce that $1 - p \geq p_C(\mathbb{H})$.

We introduce a local modification of $\mathbb{T}$, denoted by $\mathbb{T}'$, obtained by transforming a single triangle $\{u, v, w\}$ of $\mathbb{T}$ into a vertex x with three edges, as in Figure 2.

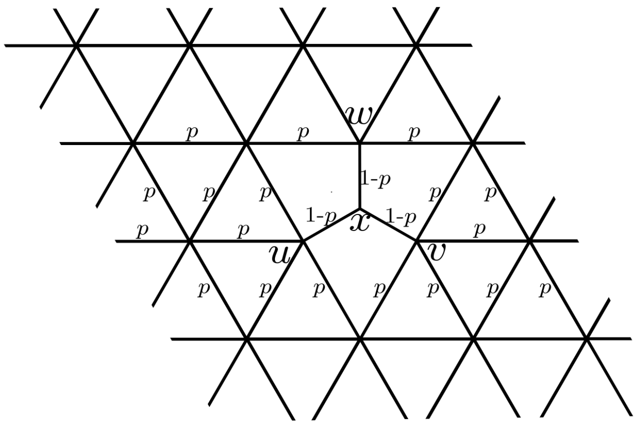


Figure 2: The modified graph $\mathbb{T}'$. Some of the bond probabilities are shown.

2. Show that if $p \in (0,1)$ satisfies $p^3 = 3p - 1$, then there exists a coupling between percolations on $\mathbb{T}$ and $\mathbb{T}'$ that preserves all connections. More precisely, argue that there exists a measure on $\{0,1\}^{E(\mathbb{T})} \times \{0,1\}^{E(\mathbb{T}')}$ such that

   - the marginal on $\{0,1\}^{E(\mathbb{T})}$ is $\mathbb{P}_p^{\mathbb{T}}$,
   - the marginal on $\{0,1\}^{E(\mathbb{T}')}$ is the percolation on $\mathbb{T}'$ with edge-probability p on all edges *except those around x that have edge-probability* $1-p$,
   - if (ω, ω') has this distribution, then a.s. for any vertices $y, z \neq x$, we have $y \leftrightarrow z$ in ω *iff* $y \leftrightarrow z$ in ω'.

From now on, we assume that p is the (unique) element of $(0,1)$ such that $p^3 = 3p - 1$.

3. Using the previous two questions, show that $p \geq p_C(\mathbb{T})$.

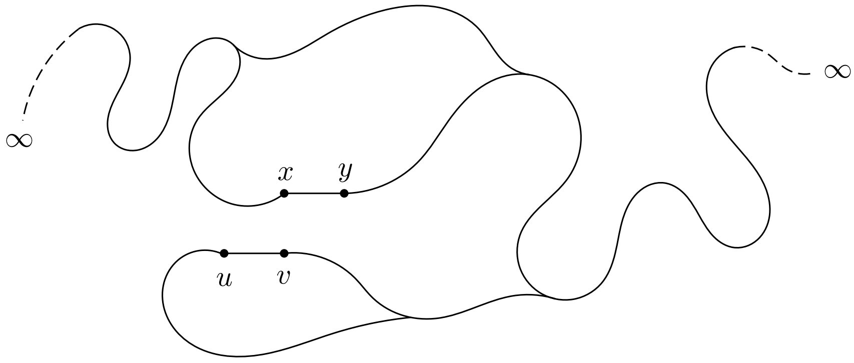
We want to prove that in fact $p = p_c(\mathbb{T})$. In the next question, we assume by contradiction that $p > p_c(\mathbb{T})$.

4. let $k < n$, and let $n' = \lfloor \frac{n-k}{2} \rfloor$, consider the event $\mathcal{B}_n$ that the boundary of $(n', n') + \Lambda_k$ is connected to the left-boundary of $\delta\Lambda_n$ *and* to the right-boundary of $\delta\Lambda_n$.

   (a) Show that $\mathbb{P}_p^{\mathbb{T}}(\mathcal{B}_n) \geq 1 - 2\mathbb{P}_p^{\mathbb{T}}(\Lambda_k \not\leftrightarrow \infty)^{1/4}$
   (b) Show that $\mathbb{P}_p^{\mathbb{T}}(\mathcal{B}_n \setminus \mathcal{A}_n) \to 0$ as $n \to \infty$.
   (c) Conclude that $\mathbb{P}_p^{\mathbb{T}}(\mathcal{A}_n) \to 1$ as $n \to \infty$.
   (d) Using duality and question 2, find a contradiction.

5. Conclude, and give also an equation satisfied by $p_c(\mathbb{H})$.

One can in fact identify that $p_c(\mathbb{T}) = 2\sin(\pi/18)$, and the apparition of a trigonometric formula here is no pure chance... Look up "isoradial graphs" for more on this subject.



The edge xy is in $\tilde{\omega}$; the edge uv is not.

Exercise 3 (Infinitely traversed edges)**.** Consider bond percolation on the square lattice graph $(\mathbb{Z}^2, \mathbb{E})$ at any percolation parameter $p \in [0,1]$. Let ω denote the random set of open edges. We say that an edge $xy \in \mathbb{E}$ is *infinitely traversed* if there is an ω-open self-avoiding path $(p_k)_{k \in \mathbb{Z}} \subset \mathbb{Z}^2$ indexed by all integers (positive and negative), such that $p_0 p_1 = xy$; see the figure above. Write $\tilde{\omega} \subset \omega$ for the random set of infinitely traversed edges. Write N and $\tilde{N}$ for the number of infinite clusters of ω and $\tilde{\omega}$. Prove that for each $p \in [0,1]$,

$$\mathbb{P}_p(N = \tilde{N}) = 1.$$

You may use any results from the course, provided that they are clearly stated.