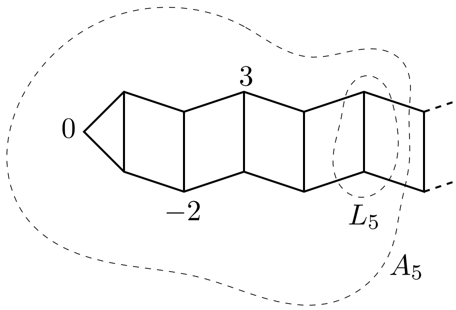
PERCOLATION
Exam 1
3 exercises; 3 pages

Exercise 1. Let $(\mathbb{Z}, E)$ denote the *ladder graph*, that is, the line graph $\mathbb{Z}$ with for each positive integer k an extra edge which connects k to $-k$. Let $L_k := \{-k, k\} \subset \mathbb{Z}$; $A_k := \cup_{\ell=0}^{k} L_\ell$. See the figure below.



Let $\mathbb{P}_p$ denote the percolation measure with percolation parameter $p \in (0,1)$ on this graph. Let C^k denote the set of vertices which are connected to 0 by an open path *which does not go outside* A_k. Define the random variables $(X_k)_{k\geq 0}$ by

$$X_k := \begin{cases} 2 & \text{if } k = 0, \\ |C^k \cap L_k| & \text{if } k > 0. \end{cases}$$

1. Argue that $(X_k)_{k\geq 0}$ is a Markov chain and find explicitly its transition matrix.
2. What is the invariant distribution of this Markov chain?
3. For fixed p, argue that $\mathbb{P}_p(0 \longleftrightarrow L_k)$ decays exponentially fast in k.
4. For fixed p, express the decay rate of $(\mathbb{P}_p(0 \longleftrightarrow L_k))_{k\geq 0}$ in terms of the spectrum of the transition matrix. (There is no need to calculate any of the eigenvalues of the transition matrix.) The decay rate of some sequence $(a_k)_{k\geq 0} \subset [0,1]$ is defined as

$$\sup\{\alpha \geq 0 : \exists C > 0, \forall k, a_k \leq Ce^{-\alpha k}\}.$$

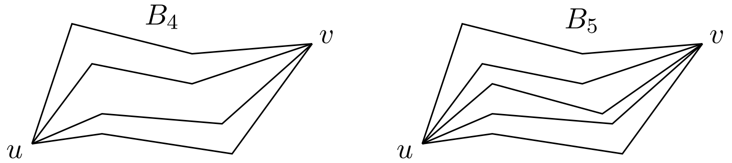
Exercise 2. 1. For a given simple countable vertex-transitive graph G, recall the definition of $\theta(p)$, and also the definition of $p_c(G)$.

2. Recall the definition of the hypercubic lattice $(\mathbb{Z}^d, \mathbb{E}^d)$ in any dimension $d \geq 1$.
3. Give a brief explanation why $(p_c((\mathbb{Z}^d, \mathbb{E}^d)))_{d\geq 0}$ is non-increasing in d.
4. Prove that $p_c((\mathbb{Z}^2, \mathbb{E}^2)) \leq \frac{3}{4}$.

The next objective in this exercise is to prove that

$$\lim_{d\to\infty} p_c((\mathbb{Z}^d, \mathbb{E}^d)) = 0. \tag{1}$$

We will do this in several steps. For each $k \geq 0$, write B_k for the following graph (illustrated also by the figure below). There are two distinct distinguised vertices u and v. They are connected by k "bridges", which are simply three edges linked in series. Thus, the graph has $2k+2$ vertices and $3k$ edges in total.



5. Write down an explicit formula for $b(p,k) := \mathbb{P}_p^{B_k}(u \longleftrightarrow v)$, where $\mathbb{P}_p^G$ denotes percolation on the graph G.

If you do not manage to find this formula, you may use in the remainder of the exercise that for any $p \in (0,1]$, we have $\lim_{k\to\infty} b(p,k) = 1$.

Let $\{a \overset{3}{\longleftrightarrow} b\}$ denote the event that two vertices a and b are connected through an open path of length exactly 3.

6. Let $a, b \in \mathbb{Z}^d$ denote two neighbours in the hypercubic lattice graph. Show that

$$\mathbb{P}_p^{(\mathbb{Z}^d,\mathbb{E}^d)}(a \overset{3}{\longleftrightarrow} b) \geq b(p, 2d-2).$$

Write $(Z^{2,d}, E^{2,d})$ for the embedded two-dimensional square lattice graph defined by

$$Z^{2,d} := \mathbb{Z}^2 \times \{0\}^{d-2} \subset \mathbb{Z}^d; \qquad E^{2,d} := \{e \in \mathbb{E}^d : e \subset Z^{2,d}\}.$$

If the events $(\{a \overset{3}{\longleftrightarrow} b\})_{ab\in E^{2,d}}$ were independent, then we could prove that

$$b(p, 2d-2) \geq \frac{3}{4} \implies p_c((\mathbb{Z}^d, \mathbb{E}^d)) \leq p$$

by arguing exactly as for the square lattice.

7. Argue that the events $(\{a \overset{3}{\longleftrightarrow} b\})_{ab\in E^{2,d}}$ are not independent.
8. Formulate a simple criterion for a family of percolation events to be independent. No proof is required.
9. Modify slightly each event $\{a \overset{3}{\longleftrightarrow} b\}$, so that the modified family is a family of independent events. Use this modified family to show that

$$b(p, \lfloor \frac{2d-4}{4} \rfloor) \geq \frac{3}{4} \implies p_c((\mathbb{Z}^d, \mathbb{E}^d)) \leq p.$$

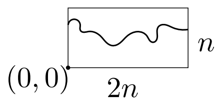
   Conclude that the limit in Equation (1) holds true.
10. Does $(p_c((\mathbb{Z}^d, \mathbb{E}^d)))_d$ tend to zero exponentially fast? In your answer, you are allowed to recall results from the lectures without giving a proof.

Exercise 3. 1. Recall the precise statement of the Harris inequality. A proof is not required.

2. Prove the square root trick: if $\mathbb{P}_p$ denotes the percolation measure on some graph G, and $(A_k)_{1\le k\le n}$ some finite family of increasing events, then

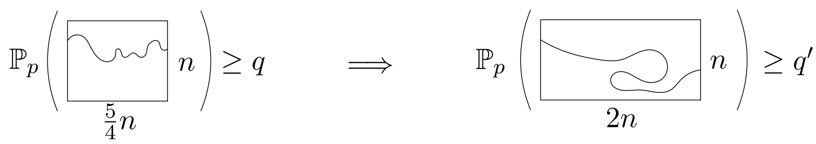
$$\max_k \mathbb{P}_p(A_k) \ge 1 - \sqrt[n]{1 - \mathbb{P}_p(A_1 \cup \cdots \cup A_n)}.$$

In the remainder of the exercise, we work on the square lattice graph $G = (\mathbb{Z}^2, \mathbb{E}^2)$. We use figures to denote crossing events in the standard way, e.g., the figure

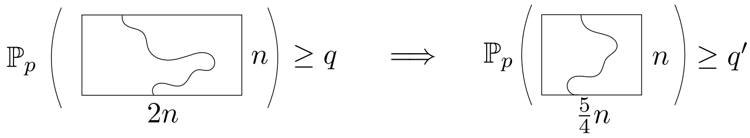


denotes the event that there is an open path from $\{0\} \times [0, n]$ to $\{2n\} \times [0, n]$ whose vertices lie in $[0, 2n] \times [0, n]$. The probability of such an event does not depend on the position of the lower-left corner of the rectangle.

3. Prove that for each $q > 0$, there exists a $q' > 0$ such that for any positive integer multiple n of 24000, we have



4. Prove that for each $q > 0$, there exists a $q' > 0$ such that for any positive integer multiple n of 24000, we have



5. Prove that in the previous exercise, if q is close to one, then q' may also be taken close to one. In other words, show that one may choose $q' := f(q)$ where f is some function with

$$\lim_{q\uparrow 1} f(q) = 1.$$