MIRROR SYMMETRY IN PERCOLATION

UNIVERSITÉ PARIS-SACLAY
MASTER2 PROJECT PROPOSAL
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Consider percolation on the square lattice $(\mathbb{Z}^2, \mathbb{E})$, and consider mirror symmetry through the y-axis. Pick two points $a, b \in \mathbb{Z}^2$ on the left of the y-axis, and let $a', b' \in \mathbb{Z}^2$ denote their reflected counterparts. Then a is closer to b than to b' in any reasonable metric (such as the graph metric, the Euclidean distance, et cetera). In particular, we expect that

$$\mathbb{P}_p \left(a \xleftrightarrow{\omega} b \right) \geq \mathbb{P}_p \left(a \xleftrightarrow{\omega} b' \right) \tag{1}$$

for any $p \in [0, 1]$. However, such inequalities remain mysteriously open.

The Ising model is a model of small magnets, each valued ± 1. The sample space is thus the set $\Omega = \{-1, 1\}^{\mathbb{Z}^2}$. There is a parameter $\beta > 0$, called the *inverse temperature*. The probability measure $\langle \cdot \rangle_\beta$ is such that neighbouring spins prefer to align. The well-known inequality of Messager and Miracle-Solé says that

$$\langle \sigma_a \sigma_b \rangle_\beta \geq \langle \sigma_a \sigma_{b'} \rangle_\beta.$$

This is precisely the Ising counterpart to (1). This suggests that the Ising model is easier to study on the one hand, but also that our intuition of *being closer* makes sense in statistical mechanics, thus providing further evidence for the conjecture in the context of percolation.

Moreover, (1) is known for percolation on certain specific graphs: for example, start with a complete graph on the left and right of the y-axis, and connect each vertex to the corresponding vertex on the other side of the reflection line.

This problem in percolation theory is generally known as the *bunkbed conjecture* and was first proposed by Kasteleyn.

The ultimate objective of this research project is to understand why the problem is so difficult, and to expand the number of graphs for which the inequality is known. A succesful thesis may include:

- A modern presentation of the proof of the Messager-Miracle-Solé inequality,
- An overview of the bunkbed conjecture, including proofs for specific graphs or reductions of the problem,
- A proof of the bunkbed conjecture for new graphs,
- Other directions proposed by the student.

We list a few references. For a general overview: [Häg03, Lin11]. For specific graphs: for large values of p [HNNK21], for the complete graph [dB16, dB18, vHL19], for the Ising model: original paper [MMS77], modern presentation [DC17].

References

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