Diffusivity of a walk on fracture loops of a discrete torus

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18 June 2017

Abstract

In this paper we study functions on the discrete torus which have a crystalline structure. This means that if we fix such a function and walk around the torus in a positive direction, the function increases on almost every step, except at a small number of steps where it must go down in order to meet the periodicity of the torus. It turns out that the down steps are organised into a small number of closed simple disjoint paths, the fracture lines of the crystal. We define a random walk on the resulting functions, the law of which is Brownian in the diffusive limit. We show that in the limit of the crystal structure becoming microscopic, the diffusivity is given by $\sigma^2 = (1 + 2\gcd(\mathbf{n}_1, \mathbf{n}_2))^{-1}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the number of fractures in the horizontal and vertical direction respectively. This is the main result of this paper. The diffusivity of the corresponding one-dimensional model has already been studied by Espinasse, Guillotin-Plantard and Nadeau, and this paper generalises that model to two dimensions. However, the methodology involving an analysis of the fracture lines that we use to calculate the diffusivity is completely novel.

1. Introduction and results

We first define **pn**-periodic height functions. The model depends on the choice of a dimension, d (we will later take $d = 2$), and on the choice of two elements $\mathbf{p}, \mathbf{n} \in \mathbb{N}^d$. We will soon give meaning to these numbers. Fix a triple $(d, \mathbf{p}, \mathbf{n})$ and define $\mathbf{t} := \mathbf{p}+\mathbf{n}$. Define the **t**-periodic lattice to be the graph with vertex set $V_\mathbf{t} := \prod_{i=1}^d (\mathbb{Z}/\mathbf{t}_i\mathbb{Z})$ and edge set $E_\mathbf{t} := \{\{\mathbf{x}, \mathbf{x}+e_i\} : \mathbf{x} \in V_\mathbf{t}, 1 \leq i \leq d\}$. Write $\mathbf{0}$ for the naturally distinguished zero element of the set $V_\mathbf{t}$. We identify the edge $\{\mathbf{x}, \mathbf{x} + e_i\}$ with the pair $(\mathbf{x}, i)$.

Definition 1.1. A **pn**-periodic height function is a map $f : V_\mathbf{t} \to \mathbb{R}$ that satisfies

$$f(\mathbf{x} + e_i) - f(\mathbf{x}) \in \left\{ -\frac{\mathbf{p}_i - \mathbf{n}_i}{\mathbf{t}_i} + 1, -\frac{\mathbf{p}_i - \mathbf{n}_i}{\mathbf{t}_i} - 1 \right\}$$

for all $(\mathbf{x}, i) \in E_\mathbf{t}$. A **pn**-periodic height function will also be called a height function.

A **pn**-periodic height function may be thought of as assigning a height to each vertex in the lattice. In Figure 1 we see a picture of a height function that shows the fracture

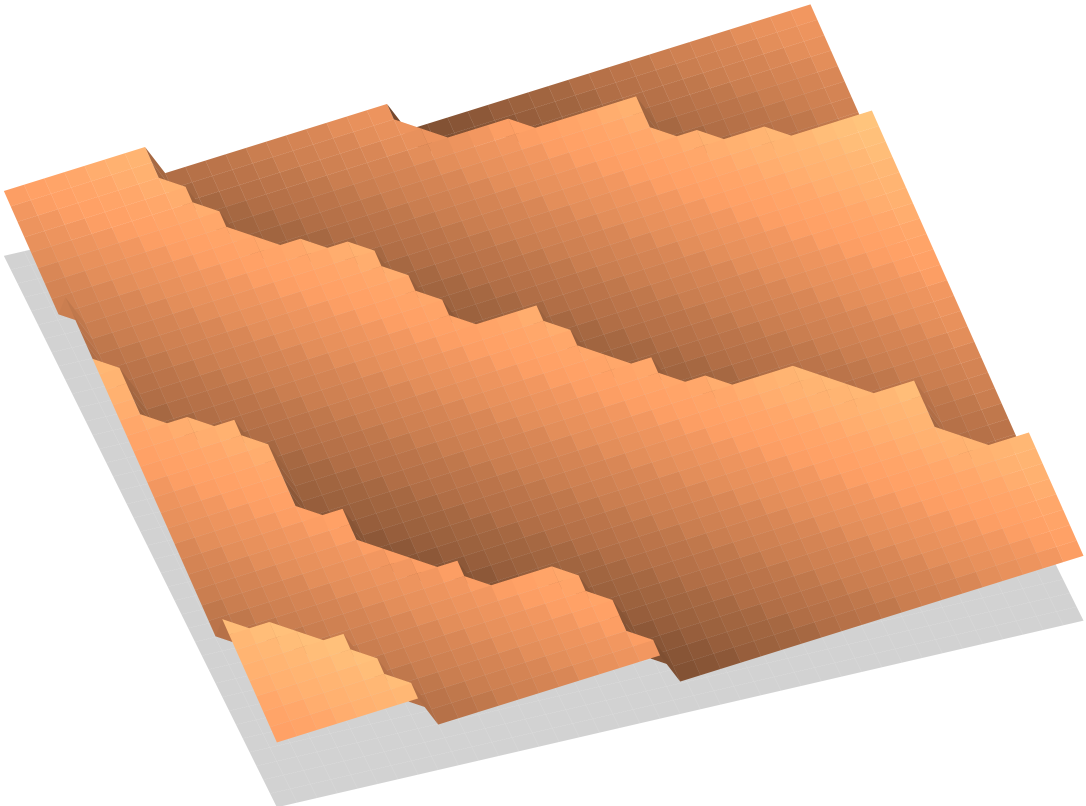

Figure 1: An example of a $(38,38)(2,2)$-periodic height function. The domain of this function is a (discrete) torus, so the opposite sides of the square have been identified. We see that if we start at a point on the torus and walk into one direction then we step down exactly twice before returning to the same point. This picture already suggests that the down steps are organised into (in this case two) fracture loops that wind around the torus.

lines that we study later. An example of a height function with explicit values is given in Figure 2a. Let f be a height function. To understand the definition of a height function and the meaning of $\mathbf{p}$ and $\mathbf{n}$, it is helpful to introduce the following map. Define the map $\operatorname{sgn}_f : E_{\mathbf{t}} \to \{-1, 1\}$ by

$$f(\mathbf{x} + e_i) - f(\mathbf{x}) = -\frac{\mathbf{p}_i - \mathbf{n}_i}{\mathbf{t}_i} + \operatorname{sgn}_f(\mathbf{x}, i). \tag{1}$$

Say that $\operatorname{sgn}_f(\mathbf{x}, i)$ is the sign of the edge $(\mathbf{x}, i)$ with respect to f, and if $\operatorname{sgn}_f(\mathbf{x}, i) = -1$ then we call the edge $(\mathbf{x}, i)$ a down step of f (otherwise the edge is an up step of f). Fix, for this moment, a reference point $\mathbf{x} \in V_{\mathbf{t}}$ and a direction $1 \leq i \leq d$. There are $\mathbf{t}_i$ edges of the form $(\mathbf{x} + ke_i, i)$. By summing (1) over these edges, we obtain

$$\sum\nolimits_{k=0}^{\mathbf{t}_i - 1} \big(f(\mathbf{x} + ke_i + e_i) - f(\mathbf{x} + ke_i)\big) = -(\mathbf{p}_i - \mathbf{n}_i) + \sum\nolimits_{k=0}^{\mathbf{t}_i - 1} \operatorname{sgn}_f(\mathbf{x} + ke_i, i). \tag{2}$$

The terms of the sum on the left collapse, so both sides of this equation equal zero. Therefore, in this collection of $\mathbf{t}_i$ edges, $\mathbf{p}_i$ edges are up steps of f, and $\mathbf{n}_i$ edges are down steps of f. In other words, if we start at $\mathbf{x}$ and step into direction i until we return to $\mathbf{x}$, then f increases along $\mathbf{p}_i$ such steps, and f decreases along $\mathbf{n}_i$ steps. Write $\hat{f}$ for the average height of f, i.e.,

$$\hat{f} := |V_{\mathbf{t}}|^{-1} \sum\nolimits_{\mathbf{x} \in V_{\mathbf{t}}} f(\mathbf{x}).$$

In this paper we study walks on height functions. Call two height functions f and g neighbours if $f(\mathbf{x}) - g(\mathbf{x}) \in \{-1, 1\}$ for all $\mathbf{x} \in V_{\mathbf{t}}$, and write $f \sim g$ (this is not an equivalence relation). The number of neighbours of a height function is bounded by $2^{|V_{\mathbf{t}}|}$, hence finite. Therefore we can consistently define a random walk $X^{\mathbf{pn}} = (X_n^{\mathbf{pn}})_{n \geq 0}$ on $\mathbf{pn}$-periodic height functions. We impose that $X_0^{\mathbf{pn}}(\mathbf{0}) \in \mathbb{Z}$, so that $X_n^{\mathbf{pn}}(\mathbf{0}) \in \mathbb{Z}$ for all $n \geq 0$. Say that two height functions f and g have the same shape if they differ by a constant. We see in the next section that the number of distinct shapes is finite. The walk $X^{\mathbf{pn}}$ turns out to be irreducible. Therefore by arguments similar to those presented in [1, 2], the law of

$$\left(n^{-1/2} \hat{X}^{\mathbf{pn}}_{\lfloor nt \rfloor}\right)_{t \in [0,1]}$$

converges to that of a Brownian motion of some variance $\sigma^2(\hat{X}^{\mathbf{pn}})$ as $n \to \infty$. Espinasse et al. provide an explicit formula for $\sigma^2(\hat{X}^{\mathbf{pn}})$ in [2] if $d = 1$. The following theorem, which is the main result of this paper, addresses the case $d = 2$.

Theorem 1.2. *If $d = 2$ then*

$$\lim_{\mathbf{p} \to \infty} \sigma^2(\hat{X}^{\mathbf{pn}}) = (1 + 2 \gcd \mathbf{n})^{-1},$$

where we write $\mathbf{p} \to \infty$ for $\mathbf{p}_1, \mathbf{p}_2 \to \infty$ and $\gcd \mathbf{n}$ for $\gcd(\mathbf{n}_1, \mathbf{n}_2)$.

This theorem is interesting for the following reasons. In the case $d = 1$ the diffusivity $\sigma^2(\hat{X}^{\mathbf{pn}})$ of the walk depends on two parameters $\mathbf{p}, \mathbf{n} \in \mathbb{N}$; [2] provides an explicit expression for $\sigma^2(\hat{X}^{\mathbf{pn}})$. The value of $\sigma^2(\hat{X}^{\mathbf{pn}})$ is decreasing in both $\mathbf{p}$ and $\mathbf{n}$. In the case $d = 2$ an equivalent statement does not hold. If both components of $\mathbf{p} \in \mathbb{N}^2$ are sufficiently large and the components of $\mathbf{n} \in \mathbb{N}^2$ are sufficiently small, then the theorem tells us that the diffusivity $\sigma^2(\hat{X}^{\mathbf{pn}})$ depends mostly on $\gcd \mathbf{n}$, so increasing $\mathbf{n}_1$ will either increase or decrease $\sigma^2(\hat{X}^{\mathbf{pn}})$, depending on how the increase in $\mathbf{n}_1$ changes the value of $\gcd \mathbf{n}$. The number $\gcd \mathbf{n}$ will appear naturally in the analysis of shapes of $\mathbf{pn}$-periodic height functions as the number of loops that are required to describe such shapes. In this paper we fix $d = 2$. The element $\mathbf{n}$ will always be fixed, and $\mathbf{p}$ will usually be fixed unless we explicitly take a limit.

We now give an overview of the proof. In proving the theorem we face two challenges. First, the average height of the walk, $\hat{X}^{\mathbf{pn}}$, is not a martingale. This is different from the $d = 1$ case in [2]. Secondly, the combinatorial structure of the set of shapes is quite nontrivial, which prevents us from calculating $\sigma^2(\hat{X}^{\mathbf{pn}})$ here exactly. We instead make approximations with vanishing errors as $\mathbf{p}$ grows. The analysis consists of several parts. In Section 2 we show that the set of shapes is in bijection with a set with a nice combinatorial structure and we show that the walk $X^{\mathbf{pn}}$ is irreducible. In Section 3 we show that every shape is naturally related to a collection of $\gcd \mathbf{n}$ monotone loops that are embedded in $C_{\mathbf{t}} := (\mathbb{R}/\mathbf{t}_1\mathbb{Z}) \times (\mathbb{R}/\mathbf{t}_2\mathbb{Z})$, the continuous torus which contains the discrete lattice $V_{\mathbf{t}}$. In Section 4 we study pairs of shapes; we determine under what conditions two shapes are neighbours, and introduce a way to calculate the difference in average height between certain pairs of shapes. In Section 5 we show that a monotone loop that is chosen uniformly at random is likely to be close to a diagonal line if the components of $\mathbf{p}$ are large. The slope of this diagonal is determined by $\mathbf{n}$ and $\mathbf{t} = \mathbf{p} + \mathbf{n}$. Consequently in the probabilistic setting the behavior of a randomly selected loop is almost completely determined by its starting point since this starting point determines the diagonal that the loop remains close to. This allows us to reduce to a one-dimensional model. In the limit we further reduce to a continuous model. In Section 6 we find a martingale that approximates the random process $\hat{X}^{\mathbf{pn}}$, which is itself not a martingale. In Section 7 we show that all approximation errors go to zero as $\mathbf{p} \to \infty$, and we prove the main result.

2. Introduction of shapes

If f is a height function then we define $[f]$, the shape of f, to be the equivalence class of height functions with the same shape, so

$$[f] = \{..., f - 2, f - 1, f, f + 1, f + 2, ...\},$$

and we write $\mathcal{S}_{\mathbf{pn}}$ for the set of shapes. The main goal of this section is to prove that the set of shapes $\mathcal{S}_{\mathbf{pn}}$ is in natural bijection with some set $\mathcal{N}_{\mathbf{tn}}$ that has a nice

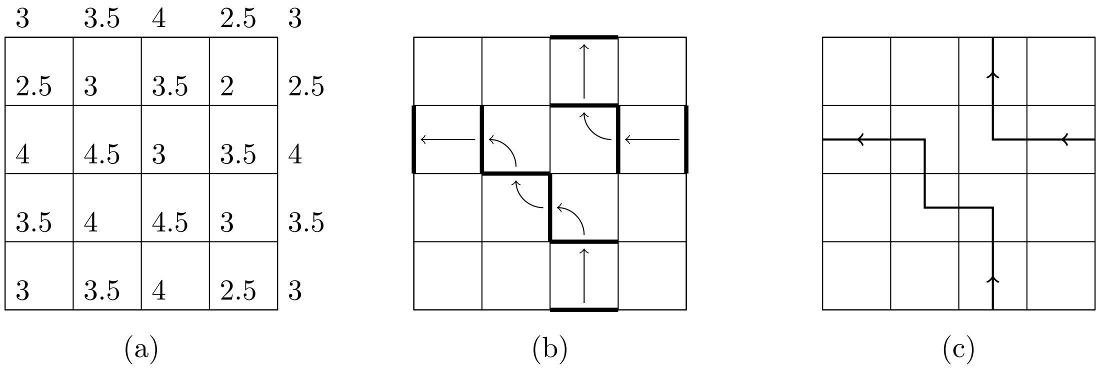

Figure 2: An example of a $(3,3)(1,1)$-periodic height function f. In subfigure (a) we see the explicit values that f takes on the set $V_{(4,4)}$. This set is periodic; opposing sides are identified. In subfigure (b) the edges in $\nu(f)$ have been thickened. The arrows demonstrate how π^A_{12} permutes the elements of $A = \nu(f)$. Subfigure (c) displays the loop in the corresponding set $\tilde{\mathcal{F}}$. Since $\gcd(1,1) = 1$ this set contains precisely one loop.

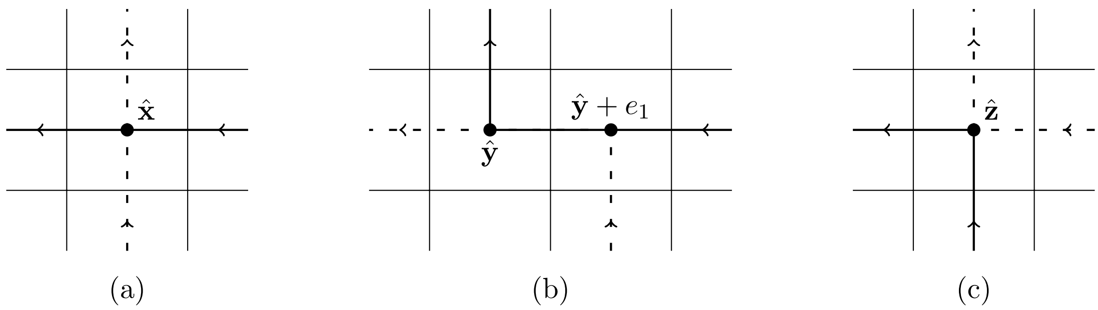

Figure 3: Three way in which two loops α^1 (continuous line) and α^2 (dashed line) in $\mathcal{K}_{\mathbf{td}}$ can intersect. If $\alpha^1, \alpha^2 \in \tilde{\mathcal{F}}$ where $\tilde{\mathcal{F}}$ is constructed from a shape $A \in \mathcal{N}_{\mathbf{tn}}$ then intersections (a) and (b) cannot occur. An intersection of type (a) implies that π^A_{12} maps the edge $(\mathbf{x}+e_1, 2)$ to $(\mathbf{x}, 2)$ and the edge $(\mathbf{x}, 1)$ to $(\mathbf{x}+e_2, 1)$. This contradicts the definition of π^A_{12} as all four edges are contained in A. An intersection of type (b) implies that π^A_{12} maps the edges $(\mathbf{y} + 2e_1, 2)$ and $(\mathbf{y} + e_1, 1)$ to $(\mathbf{y} + e_1, 2)$. This contradicts that π^A_{12} is a bijection. An intersection of type (c) is called a touch.

combinatorial structure. This is Lemma 2.1. To prove that $\hat{X}^{\mathbf{pn}}$ is Brownian under diffusive scaling it is necessary to prove that the random walk $X^{\mathbf{pn}}$ is irreducible, which is Lemma 2.2. This lemma plays no further role in our analysis.

Let f and g be two height functions. Note that f and g have the same shape if and only if $\mathrm{sgn}_f(\mathbf{x}, i) = \mathrm{sgn}_g(\mathbf{x}, i)$ for all $(\mathbf{x}, i) \in E_{\mathbf{t}}$. Therefore the number of shapes is bounded by $2^{|E_{\mathbf{t}}|}$, hence finite. For the same reason the map

$$\nu : \mathcal{S}_{\mathbf{pn}} \to \mathcal{P}\left(E_{\mathbf{t}}\right),\ [f] \mapsto \{(\mathbf{x}, i) \in E_{\mathbf{t}} : \mathrm{sgn}_f(\mathbf{x}, i) = -1\}$$

is well-defined and injective. In Figure 2b we see what the set $\nu(f)$ looks like (we will write $\nu(f)$ for $\nu([f])$). So the set $\nu(f) \subset E_{\mathbf{t}}$ contains the edges with negative sign with respect to f; it is the set of down steps of f. Define for convenience $\mathbf{q}_i := \frac{\mathbf{p}_i - \mathbf{n}_i}{\mathbf{t}_i}$ and let $\mathbf{x} \in V_{\mathbf{t}}$. By (1),

$$\begin{aligned} f(\mathbf{x} + e_1 + e_2) &= f(\mathbf{x}) + \mathrm{sgn}_f(\mathbf{x}, 1) + \mathrm{sgn}_f(\mathbf{x} + e_1, 2) - \mathbf{q}_1 - \mathbf{q}_2 \\ &= f(\mathbf{x}) + \mathrm{sgn}_f(\mathbf{x}, 2) + \mathrm{sgn}_f(\mathbf{x} + e_2, 1) - \mathbf{q}_1 - \mathbf{q}_2, \end{aligned}$$

from which we deduce that

$$\mathrm{sgn}_f(\mathbf{x}, 1) + \mathrm{sgn}_f(\mathbf{x} + e_1, 2) = \mathrm{sgn}_f(\mathbf{x}, 2) + \mathrm{sgn}_f(\mathbf{x} + e_2, 1). \tag{3}$$

Say that a set $A \subset E_{\mathbf{t}}$ satisfies the square condition (s.c.) at $\mathbf{x} \in V_{\mathbf{t}}$ if

$$|A \cap \{(\mathbf{x}, 1), (\mathbf{x} + e_1, 2)\}| = |A \cap \{(\mathbf{x}, 2), (\mathbf{x} + e_2, 1)\}|.$$

Say that A satisfies the square condition if it satisfies the square condition at every element $\mathbf{x} \in V_{\mathbf{t}}$. It follows from (3) that the set $\nu(f)$ satisfies the square condition. This observation is key to the analysis of shapes of height functions. Write $L_{(\mathbf{x},i)}$ for the set $\{(\mathbf{x} + ke_i, i) \in E_t : k \in \mathbb{Z}\}$. We may think of $L_{(\mathbf{x},i)}$ as the line starting from $\mathbf{x}$ in direction i; the set $L_{(\mathbf{x},i)}$ contains $\mathbf{t}_i$ edges. Exactly $\mathbf{n}_i$ edges in $L_{(\mathbf{x},i)}$ are down steps of f by (2) and the remark that follows that equation. Hence $|\nu(f) \cap L_{(\mathbf{x},i)}| = \mathbf{n}_i$. The two observations we made about $\nu(f)$ motivate us to define

$$\mathcal{N}_{\mathbf{ta}} := \{A \subset E_{\mathbf{t}} : A \text{ satisfies the s.c. and } |A \cap L_{(\mathbf{x},i)}| = \mathbf{a}_i \text{ for any } (\mathbf{x}, i) \in E_{\mathbf{t}}\},$$

for any $\mathbf{a}$ with $\mathbf{a}_i \in \{0, 1, ...\mathbf{t}_i\}$. By the two observations we see that $\nu(f)$, set of down steps of f, is an element of $\mathcal{N}_{\mathbf{tn}}$.

Lemma 2.1. *The map ν is a bijection from $\mathcal{S}_{\mathbf{pn}}$ to $\mathcal{N}_{\mathbf{tn}}$.*

We will write $ae_1 + be_2$ for the point $\mathbf{0} + ae_1 + be_2 \in V_{\mathbf{t}}$ when no confusion is likely to arise.

Proof. Let $A \in \mathcal{N}_{\mathbf{tn}}$. We will construct a **pn**-periodic height function f with $\nu(f) = A$. For this function f it is necessary and sufficient to satisfy $f(\mathbf{0}) \in \mathbb{Z}$ and

$$f(\mathbf{x} + e_i) - f(\mathbf{x}) = -\mathbf{q}_i + 1_{(\mathbf{x},i)\notin A} - 1_{(\mathbf{x},i)\in A} = -\mathbf{q}_i + 1 - 2 \cdot 1_{(\mathbf{x},i)\in A} \tag{4}$$

for any $(\mathbf{x}, i) \in E_{\mathbf{t}}$. We inductively define the map $f : V_{\mathbf{t}} \to \mathbb{R}$ by $f(\mathbf{0}) = 0$,

$$f(ke_1 + e_1) := f(ke_1) - \mathbf{q}_1 + 1 - 2 \cdot 1_{(ke_1,1)\in A}$$

for all $0 \leq k < \mathbf{t}_1 - 1$ and

$$f(ke_1 + le_2 + e_2) := f(ke_1 + le_2) - \mathbf{q}_2 + 1 - 2 \cdot 1_{(ke_1+le_2,2)\in A}$$

for all $0 \leq k < \mathbf{t}_1$ and $0 \leq l < \mathbf{t}_2 - 1$. We first show that (4) holds for all pairs $(\mathbf{x}, i)$ with $\mathbf{x}_i \neq \mathbf{t}_i - 1$. For $i = 2$ this follows from the definition of f, so consider $i = 1$. We need to show that for all $0 \leq k < \mathbf{t}_1 - 1$ and for all $0 \leq l < \mathbf{t}_2$,

$$f(ke_1 + le_2 + e_1) := f(ke_1 + le_2) - \mathbf{q}_1 + 1 - 2 \cdot 1_{(ke_1+le_2,1)\in A}.$$

We induct on l; the case $l = 0$ follows from the definition of f. For $l > 0$,

$$\begin{aligned} f(ke_1 + le_2 + e_1) - f(ke_1 + le_2) &= f(ke_1 + (l-1)e_2 + e_1) - f(ke_1 + (l-1)e_2) \\ &\quad - 2(1_{(ke_1+(l-1)e_2+e_1,2)\in A} - 1_{(ke_1+(l-1)e_2,2)\in A}) \\ &= -\mathbf{q}_1 + 1 - 2 \cdot 1_{(ke_1+(l-1)e_2,1)\in A} \\ &\quad - 2(1_{(ke_1+(l-1)e_2+e_1,2)\in A} - 1_{(ke_1+(l-1)e_2,2)\in A}) \\ &= -\mathbf{q}_1 + 1 - 2 \cdot 1_{(ke_1+le_2,1)\in A}, \end{aligned}$$

where the last equality is due to A satisfying the square condition. This completes the induction argument. We have shown that (4) holds for all pairs $(\mathbf{x}, i)$ with $\mathbf{x}_i \neq \mathbf{t}_i - 1$. For any pair $(\mathbf{x}, i)$ with $\mathbf{x}_i = \mathbf{t}_i - 1$, we have

$$\begin{aligned} f(\mathbf{x} + e_i) - f(\mathbf{x}) &= -\sum_{k=1}^{\mathbf{t}_i-1} \left(f(\mathbf{x} + ke_i + e_i) - f(\mathbf{x} + ke_i)\right) \\ &= -(-\mathbf{q}_i + 1)(\mathbf{t}_i - 1) + 2\sum_{k=1}^{\mathbf{t}_i-1} 1_{(\mathbf{x}+ke_i,i)\in A} \\ &= -\mathbf{q}_i + 1 - 2\mathbf{n}_i + 2\sum_{k=1}^{\mathbf{t}_i-1} 1_{(\mathbf{x}+ke_i,i)\in A} \\ &= -\mathbf{q}_i + 1 - 2 \cdot 1_{(\mathbf{x},i)\in A}, \end{aligned}$$

where the last equality is due to the assumption that $|A \cap L_{(\mathbf{x},i)}| = \mathbf{n}_i$. $\square$

Lemma 2.2. *A random walk on* **pn***-periodic height functions is irreducible.*

Proof. We show that the graph on height functions is connected. If f and g have the same shape then it is clear that f and g are connected (since f is a neighbour of $f+1$, it is also connected with $f+2$, etc.). For any height function f we define $\chi(f)=\sum_{\mathbf{x}\in V_{\mathbf{t}}}(f(\mathbf{x})-f(\mathbf{0}))$, which is constant on shapes. Let

$$A^* := \{(\mathbf{x},i)\in E_{\mathbf{t}} : \mathbf{x}_i \in \{0,1,...,\mathbf{n}_i-1\}\} \in \mathcal{N}_{\mathbf{tn}}.$$

Suppose that $\nu(f)\neq A^*$. It suffices to demonstrate that there exists a neighbour g of f with $\chi(g)<\chi(f)$. We claim that there exists an element $\mathbf{z}\in V_{\mathbf{t}}\setminus\{\mathbf{0}\}$ such that $(\mathbf{z},1),(\mathbf{z},2)\in\nu(f)$ and such that $(\mathbf{z}-e_1,1),(\mathbf{z}-e_2,2)\notin\nu(f)$. If $\nu(f)\neq A^*$ then there exists an edge $(\mathbf{y},i)\in\nu(f)$ such that $\mathbf{y}_i\neq 0$ and $(\mathbf{y}-e_i,i)\notin\nu(f)$. Without loss of generality we assume that $i=1$. Let n be the smallest nonnegative integer such that $(\mathbf{y}+ne_2,2)\in\nu(f)$ and let m be the smallest nonnegative integer such that $(\mathbf{y}-(m+1)e_2,2)\notin\nu(f)$. We choose $\mathbf{z}=\mathbf{y}+ne_2$ if $n>0$ and we choose $\mathbf{z}=\mathbf{y}-me_2$ if $n=0$. We observe that in both cases $\mathbf{z}_1=\mathbf{y}_1\neq 0$, so that $\mathbf{z}\neq\mathbf{0}$. By induction arguments, the square condition, minimality of n and m and the fact that $(\mathbf{y},1)\in\nu(f)$ and $(\mathbf{y}-e_1,1)\notin\nu(f)$, we deduce that

1. $(\mathbf{y}+ke_2,1)\in\nu(f)$ for all $0\leq k\leq n$,
2. $(\mathbf{y}-ke_2,1)\in\nu(f)$ for all $0\leq k\leq m$,
3. $(\mathbf{y}+ke_2-e_1,1)\notin\nu(f)$ for all $0\leq k\leq n$,
4. $(\mathbf{y}-ke_2-e_1,1)\notin\nu(f)$ for all $0\leq k\leq m$.

We provide details for the first of these four statements. We chose $\mathbf{y}$ such that the statement holds for $k=0$. Now suppose that $(\mathbf{y}+ke_2,1)\in\nu(f)$ for some $0\leq k<n$. Note that $(\mathbf{y}+ke_2,2)\notin\nu(f)$ by minimality of n. Since $\nu(f)$ satisfies the square condition (at $\mathbf{y}+ke_2$), we deduce that $(\mathbf{y}+(k+1)e_2,1)\in\nu(f)$. This proves the first statement; the others follow similarly. We have found a point $\mathbf{z}\neq\mathbf{0}$ such that $(\mathbf{z},1),(\mathbf{z},2)\in\nu(f)$ and $(\mathbf{z}-e_1,1),(\mathbf{z}-e_2,2)\notin\nu(f)$. This proves our claim. Define the height function g by $g(\mathbf{x})=f(\mathbf{x})+1-2\cdot 1_{\mathbf{x}=\mathbf{z}}$ for all $\mathbf{x}\in V_{\mathbf{t}}$ and deduce that $\chi(g)=\chi(f)-2$. By construction g is a neighbour of f. This finishes the proof of this lemma. □

3. Introduction of walks and loops

We now formally recover for each height function the corresponding set of fracture loops whose existence was suggested by Figure 1. Suppose that f is a height function, so that $\nu(f)\in\mathcal{N}_{\mathbf{tn}}$. Recall from the previous section that the set $\nu(f)$ can be any subset of $E_{\mathbf{t}}$ that satisfies the square condition and the condition $|\nu(f)\cap L_{(\mathbf{x},i)}|=\mathbf{n}_i$ for all $(\mathbf{x},i)\in E_{\mathbf{t}}$. Set $A=\nu(f)\in\mathcal{N}_{\mathbf{tn}}$. All that we say about A applies to sets in $\mathcal{N}_{\mathbf{ta}}$ as well, for any $\mathbf{a}$. This will be useful later. Let $(\mathbf{x},i)\in A$. Observe that the edge $(\mathbf{x},i)$ is in two "squares" of $V_{\mathbf{t}}$, they are

$$\{(\mathbf{x},i),(\mathbf{x}+e_i,j),(\mathbf{x},j),(\mathbf{x}+e_j,i)\} \text{ and } \{(\mathbf{x}-e_j,i),(\mathbf{x}+e_i-e_j,j),(\mathbf{x}-e_j,j),(\mathbf{x},i)\},$$

where $j = 2$ if $i = 1$ and vice versa. The square condition guarantees that for each of the two squares, the edge $(\mathbf{x}, i)$ can be bijectively matched to another edge in A that is in that same square. This procedure divides A into closed simple disjoint walks on edges in A. Each closed walk will be associated with a loop in the continuous torus. These loops are (almost) simple and non-intersecting, and the number of loops and the winding numbers of the loops in the torus are dictated by $\mathbf{n}$.

For $\{i, j\} = \{1, 2\}$ define the map $\pi_{ij}^A : A \to A$,

$$(\mathbf{x}, i) \mapsto \begin{cases} (\mathbf{x}, j) & \text{if } (\mathbf{x}, j) \in A \\ (\mathbf{x} + e_j, i) & \text{otherwise} \end{cases},$$
$$(\mathbf{x}, j) \mapsto \begin{cases} (\mathbf{x} - e_i + e_j, i) & \text{if } (\mathbf{x} - e_i + e_j, i) \in A \\ (\mathbf{x} - e_i, j) & \text{otherwise} \end{cases}.$$

Note that the square condition guarantees that this map is well-defined. It is straightforward to check that $\pi_{ji}^A \circ \pi_{ij}^A$ is the identity map on A. Therefore the map π_{ij}^A is a bijection from A to A with inverse map π_{ji}^A. Figure 2b shows how π_{12}^A permutes the elements of A. Let $\mathcal{F}$ be the cycle decomposition of the permutation π_{12}^A. So $\mathcal{F}$ contains closed simple walks in A that are pairwise disjoint. We have not specified the starting point of each walk, but these are not of interest to us. Define

$$\mathcal{A} := \{\operatorname{Im} a : a \in \mathcal{F}\}.$$

This is also the finest partition of A such that each member is closed under the map π_{12}^A. We call $\mathcal{A}$ the natural partition of A. Because π_{12}^A restricts to a bijection from $A \cap \{(\mathbf{x}, 1), (\mathbf{x} + e_1, 2)\}$ to $A \cap \{(\mathbf{x}, 2), (\mathbf{x} + e_2, 1)\}$, each member of $\mathcal{A}$ satisfies the square condition. Pick $a = ((\pi_{12}^A)^k(\mathbf{x}, i))_{0 \le k \le n} \in \mathcal{F}$. We now construct the continuous loop α corresponding to a. In Figure 2c we see the continuous loop corresponding to the discrete closed walk in 2b. As one observes in those figures, the continuous loop is obtained from the discrete loop by rotating each edge over a right angle and connecting the line segments so obtained. Formally this is done as follows. The lattice $V_\mathbf{t}$ is naturally embedded in $C_\mathbf{t} = (\mathbb{R}/\mathbf{t}_1\mathbb{Z}) \times (\mathbb{R}/\mathbf{t}_2\mathbb{Z})$. Introduce the edge midpoint map

$$\mu : E_\mathbf{t} \to C_\mathbf{t},\ (\mathbf{x}, i) \mapsto \mathbf{x} + e_i/2,$$

which is injective. For $k = \lfloor t \rfloor$ and $s = t - k$ define

$$\alpha_t = \begin{cases} \mu(\mathbf{y}, 1) + (s - 1/2)e_2 & \text{if } a_k = (\mathbf{y}, 1) \\ \mu(\mathbf{y}, 2) - (s - 1/2)e_1 & \text{if } a_k = (\mathbf{y}, 2) \end{cases};$$

this corresponds to the intuitive picture. Note that α only moves up and left. Write $\tilde{\mathcal{F}}$ for the set of continuous loops corresponding to discrete loops in $\mathcal{F}$. For $\mathbf{x} \in V_\mathbf{t}$ define $\hat{\mathbf{x}} := \mathbf{x} + (e_1 + e_2)/2$, and define $\hat{V_\mathbf{t}} := \{\hat{\mathbf{x}} : \mathbf{x} \in V_\mathbf{t}\}$. By definition the walks in

$\mathcal{F}$ are disjoint and simple (except at the endpoints), but the loops in $\tilde{\mathcal{F}}$ need not be disjoint and simple. A path in $\tilde{\mathcal{F}}$ may intersects itself or another path in $\tilde{\mathcal{F}}$. If it does so, this intersection must occur at a point in $\hat{V}_{\mathbf{t}}$, and both paths must make a right angle at this point, so that this intersection may be avoided by applying a small local homotopy to the loops. Such an intersection will be called a touch. This is illustrated by Figure 3.

Lemma 3.1. *Let $A \in \mathcal{N}_{\mathbf{tn}}$ and construct $\mathcal{A}$, $\mathcal{F}$ and $\tilde{\mathcal{F}}$ from A as above. Let $\mathbf{d} := \mathbf{n}/\gcd \mathbf{n}$. Let $a = (a_k)_{0 \leq k \leq n} \in \mathcal{F}$ and let $\alpha = (\alpha_t)_{t \in [0,n]}$ be the corresponding loop in $\tilde{\mathcal{F}}$. Then the following statements hold.*

1. *Let $(\mathbf{x}, i) \in E_{\mathbf{t}}$. If α hits $\{\mathbf{x} + s e_i : s \in [0,1]\}$ then it does so at half-integral times only, and at any such time $k + 1/2$ we have $\alpha_{k+1/2} = \mu(\mathbf{x}, i)$ and $a_k = (\mathbf{x}, i)$. Conversely if $a_k = (\mathbf{x}, i)$ then $\alpha_{k+1/2} = \mu(\mathbf{x}, i)$.*

2. *The restriction of α to integral times is a walk in $\hat{V}_{\mathbf{t}}$ and α is recovered from this restriction by linear interpolation.*

3. *The number n equals $\mathbf{d}_2\mathbf{t}_1 + \mathbf{d}_1\mathbf{t}_2$. For $\mathbf{d}_2\mathbf{t}_1$ distinct integers $0 \leq k < n$ we have $\alpha_{k+1} = \alpha_k - e_1$ and for $\mathbf{d}_1\mathbf{t}_2$ distinct integers $0 \leq k < n$ we have $\alpha_{k+1} = \alpha_k + e_2$.*

4. *The loops in $\tilde{\mathcal{F}}$ are simple up to touches and pairwise disjoint up to touches.*

5. *The sets $\mathcal{A}$, $\mathcal{F}$ and $\tilde{\mathcal{F}}$ contain $\gcd \mathbf{n}$ elements.*

We keep writing $\mathbf{d} = \mathbf{n}/\gcd \mathbf{n}$ in the sequel.

Proof. The first two statements follow immediately from the construction of the loop α from the loop a. Statement 4 has already been addressed. It suffices to prove 3 and 5. Let $L := (\mathbb{R}/\mathbf{t}_1\mathbb{Z}) \times \{0\} \subset C_{\mathbf{t}}$. By 1, the loops in $\tilde{\mathcal{F}}$ intersect L as often as the walks in $\mathcal{F}$ intersect $L_{(\mathbf{0},1)}$. Since every loop in $\tilde{\mathcal{F}}$ moves up or left only, every intersection with L corresponds to a wind in the vertical direction. Therefore the loops in $\tilde{\mathcal{F}}$ jointly wind around the torus $\mathbf{n}_1$ times in the vertical direction, and equivalently they jointly wind $\mathbf{n}_2$ times in the horizontal direction. Observe that the loops in $\tilde{\mathcal{F}}$ are not null-homotopic, simple up to touches and pairwise disjoint up to touches. Therefore there must be $\gcd \mathbf{n}$ loops in $\tilde{\mathcal{F}}$, and each loop must wind around the torus $\mathbf{d}_1$ times in the vertical direction and $\mathbf{d}_2$ times in the horizontal direction. This is because the two winding numbers of a simple loop in a torus must be coprime by Example 1.24 in [3]. These winding numbers and the size of the torus (given by $\mathbf{t}$) determine the numbers as in 3. □

We define $\mathcal{K}_{\mathbf{td}}$ to be the set of paths $\alpha = (\alpha_t)_{t \in [0,n]} \subset C_{\mathbf{t}}$ that satisfy Statements 2 and 3 from the previous lemma, so that $\tilde{\mathcal{F}} \subset \mathcal{K}_{\mathbf{td}}$.

Lemma 3.2. *Suppose that $\tilde{\mathcal{F}} \subset \mathcal{K}_{\mathbf{td}}$ satisfies Statements 4 and 5 from the previous lemma. Then there is a set $A \in \mathcal{N}_{\mathbf{tn}}$ such that $\tilde{\mathcal{F}}$ is obtained from A through above construction.*

Note that this lemma, together with the previous lemma, induces a bijection between the set $\mathcal{N}_{\mathbf{tn}}$ and the set of gcd $\mathbf{n}$-subsets of $\mathcal{K}_{\mathbf{td}}$ that satisfy Statement 4 from Lemma 3.1.

Proof of Lemma 3.2. We first nominate a candidate A, then prove that the loops constructed from A are those in $\tilde{\mathcal{F}}$. Define $\mathfrak{F} := \cup_{\alpha\in\tilde{\mathcal{F}}} \operatorname{Im}\alpha$. By Statement 1 in the previous lemma, we must have

$$A = \{(\mathbf{x}, i) \in E_{\mathbf{t}} : \mu(\mathbf{x}, i) \in \mathfrak{F}\}.$$

The loops in $\tilde{\mathcal{F}}$ jointly intersect $L = (\mathbb{R}/\mathbf{t}_1\mathbb{Z}) \times \{0\}$ exactly $\mathbf{d}_1 \gcd \mathbf{n} = \mathbf{n}_1$ times, and since the loops in $\tilde{\mathcal{F}}$ are simple up to touches and disjoint up to touches, we have $|A \cap L_{(\mathbf{0},1)}| = \mathbf{n}_1$. Similarly we have $|A \cap L_{(\mathbf{x},i)}| = \mathbf{n}_i$ for any $(\mathbf{x}, i) \in E_{\mathbf{t}}$. To show that $A \in \mathcal{N}_{\mathbf{tn}}$ it suffices to demonstrate that A satisfies the square condition. For any $\mathbf{x} \in V_{\mathbf{t}}$ we note that

$$\begin{aligned}
|A \cap \{(\mathbf{x}, 1), (\mathbf{x} + e_1, 2)\}| &= |\{(\alpha, k) \in \tilde{\mathcal{F}} \times \{1, ..., n\} : \alpha_{k-1/2} \in \{\mu(\mathbf{x}, 1), \mu(\mathbf{x} + e_1, 2)\}\}| \\
&= |\{(\alpha, k) \in \tilde{\mathcal{F}} \times \{1, ..., n\} : \alpha_k = \hat{\mathbf{x}}\}| \\
&= |\{(\alpha, k) \in \tilde{\mathcal{F}} \times \{1, ..., n\} : \alpha_{k+1/2} \in \{\mu(\mathbf{x}, 2), \mu(\mathbf{x} + e_2, 1)\}\}| \\
&= |A \cap \{(\mathbf{x}, 2), (\mathbf{x} + e_2, 1)\}|,
\end{aligned}$$

so A satisfies the square condition and consequently $A \in \mathcal{N}_{\mathbf{tn}}$. Let $\tilde{\mathcal{F}}'$ be the collection of continuous loops constructed from A through above procedure. It suffices to show that $\tilde{\mathcal{F}}$ equals $\tilde{\mathcal{F}}'$ up to perhaps the starting points of the loops (which do not concern us). We first claim that $\cup_{\alpha\in\tilde{\mathcal{F}}'} \operatorname{Im}\alpha = \cup_{\alpha\in\tilde{\mathcal{F}}} \operatorname{Im}\alpha =: \mathfrak{F}$. First, recall that $\cup_{\alpha\in\tilde{\mathcal{F}}'} \operatorname{Im}\alpha$ is obtained from A by rotating the edges in A by a right angle, and taking the union over those (see Figures 2b and 2c). Secondly, note that A is itself obtained from $\mathfrak{F}$ by breaking the latter set into line segments of length one and rotating each line segment by a right angle (and then putting the edges corresponding to these line segments in A). Thus, going from $\mathfrak{F}$ to $\cup_{\alpha\in\tilde{\mathcal{F}}'} \operatorname{Im}\alpha$ corresponds to rotating all those line segments by a right angle twice: we arrive at the same set. This proves the claim. Now note that there is a unique way (up to starting points) in which $\mathfrak{F}$ can be broken up into simple (up to touches) and disjoint (up to touches) loops such that each loop only moves up and left. Since $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ both satisfy this condition, they must be the same, up to the starting points of the loops. □

The construction of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ from a set $A \in \mathcal{N}_{\mathbf{tn}}$ is not unique, as it depends on the starting points of the walks and loops in the sets. Therefore we will, from now on, identify any walk $a \in \mathcal{F}$ with the walks that are equal to a up to a time change, and we identify any loop $\alpha \in \tilde{\mathcal{F}}$ with loops that are equal to α up to an integral time change. We apply this latter identification also to the loops in $\mathcal{K}_{\mathbf{td}}$. We remark here that each equivalence class under this identification contains $n = \mathbf{d}_2\mathbf{t}_1 + \mathbf{d}_1\mathbf{t}_2$ elements. For this we just need to check that any $\alpha \in \mathcal{K}_{\mathbf{td}}$ does not break down into a smaller loop that is traversed more than once. But this is clearly the case since the winding numbers of α

are $\mathbf{d}_1$ and $\mathbf{d}_2$ in the vertical and horizontal direction respectively, and those numbers are coprime. Write $\lambda(A)$ for $\tilde{\mathcal{F}}$, which is now uniquely determined from A. If f is a height function then we write $\lambda(f)$ for $\lambda(\nu(f))$.

4. Shapes and neighbours

We say that two sets $A, B \in \mathcal{N}_{\mathbf{tn}}$ are neighbours, and write $A \sim B$, if there exist height functions f and g that are neighbours with $\nu(f) = A$ and $\nu(g) = B$. Any element $A \in \mathcal{N}_{\mathbf{tn}}$ is a neighbour of itself, since any height function f is a neighbour of $f+1$ and also of $f-1$. If f and g are some given height functions, then it is straightforward to check if they are neighbours, and we have also no trouble in calculating their pointwise difference and the difference in averages over those height functions. In this section we aim to recover these numbers from only $\nu(f)$ and $\nu(g)$. Lemma 4.1 relates the pointwise difference between f and g to $\nu(f)$ and $\nu(g)$. If two elements $A, B \in \mathcal{N}_{\mathbf{tn}}$ satisfy certain conditions then Lemma 4.3 tells us if A and B are neighbours. Finally if two neighbours f and g satisfy some conditions then Lemma 4.4 expresses the average height difference $\hat{g} - \hat{f}$ in terms of the loops $\lambda(f)$ and $\lambda(g)$.

To facilitate the analysis we introduce three maps, that are straightforward to interpret intuitively. First, we define for any set $A \subset E_{\mathbf{t}}$ the map

$$\phi_A : E_{\mathbf{t}} \to A,\ (\mathbf{x}, i) \mapsto (\mathbf{x} + \min\{k \geq 0 : (\mathbf{x} + ke_i, i) \in A\}e_i, i)\,.$$

So if $(\mathbf{x}, i) \in E_{\mathbf{t}}$, then $\phi_A(\mathbf{x}, i)$ is the first edge in A that we hit when we shift $(\mathbf{x}, i)$ to the right (if $i = 1$) or upwards (if $i = 2$). Note that $\phi_A(\mathbf{x}, i) = (\mathbf{x}, i)$ if $(\mathbf{x}, i) \in A$. Observe also that ϕ_A is well-defined if and only if $L_{(\mathbf{x},i)}$ intersects A for any $(\mathbf{x}, i) \in E_{\mathbf{t}}$, but this condition will always be satisfied in this paper. The second map that we define is the map

$$\psi_A : A \to A,\ (\mathbf{x}, i) \mapsto (\mathbf{x} + \min\{k > 0 : (\mathbf{x} + ke_i, i) \in A\}e_i, i)\,,$$

and we note that (if ϕ_A is well-defined) $\psi_A(\mathbf{x}, i) = \phi_A(\mathbf{x} + e_i, i)$, so for $(\mathbf{x}, i) \in A$, the edge $\psi_A(\mathbf{x}, i)$ is the next edge in A that we meet if we shift $(\mathbf{x}, i)$ to the right (if $i = 1$) or upwards (if $i = 2$). Finally for a closed set $\mathfrak{F} \subset C_{\mathbf{t}}$ we define

$$\omega_{\mathfrak{F}} : C_{\mathbf{t}} \times \{1, 2\} \to \mathfrak{F},\ (\tilde{\mathbf{x}}, i) \mapsto \tilde{\mathbf{x}} + \inf\{s > 0 : \tilde{\mathbf{x}} + se_i \in \mathfrak{F}\}e_i.$$

Note that the map $\omega_{\mathfrak{F}}$ is well-defined if and only if the infimum that appears in the definition is finite always, which holds true if $\mathfrak{F}$ contains the image of a loop with nonzero winding numbers in both directions. We will use the maps $\omega_{\mathfrak{F}}$ as the continuous analogues of the discrete maps ϕ_A. Let $A \in \mathcal{N}_{\mathbf{ta}}$ (for $\mathbf{a}$ arbitrary now, but satisfying $\mathbf{a}_1, \mathbf{a}_2 > 0$) and $\mathfrak{F} = \cup_{\alpha \in \lambda(A)} \operatorname{Im} \alpha$. Then by Lemma 3.1, Statement 1,

$$\phi_A(\mathbf{x}, i) = \mu^{-1}(\omega_{\mathfrak{F}}(\mathbf{x}, i)) \qquad \text{for any } (\mathbf{x}, i) \in E_{\mathbf{t}} \text{ and} \tag{5}$$

$$\psi_A(\mathbf{x}, i) = \mu^{-1}(\omega_{\mathfrak{F}}(\mu(\mathbf{x}, i), i)) \quad \text{for any } (\mathbf{x}, i) \in A. \tag{6}$$

This connects the map ω with the maps ϕ and ψ.

Lemma 4.1. *Let f be a height function and let $A \in \mathcal{N}_{\mathbf{tn}}$.*

1. *If g is a neighbour of f and $\phi_{\nu(f)\cup\nu(g)}(\mathbf{x}, i) \in \nu(f) \setminus \nu(g)$, then $f(\mathbf{x}) = g(\mathbf{x}) + 1$.*
2. *If $\nu(f) = A$ then f has two neighbours g with $\nu(g) = A$. If $\nu(f) \neq A$ then f has at most one neighbour g with $\nu(g) = A$.*

Proof. Suppose that f has a down step before g, as seen from the reference point $\mathbf{x}$ and looking right (if $i = 1$) or upwards (if $i = 2$). This is precisely the hypothesis in 1. Then f must be larger than g before the down step, and smaller than g right after the down step. In particular, f must be larger than g at $\mathbf{x}$, hence $f(\mathbf{x}) = g(\mathbf{x}) + 1$. Now 2. If $\nu(f) = A$ then the two neighbours are $g = f \pm 1$. Now suppose that $\nu(f) \neq A$ and that g is a neighbour of f with $\nu(g) = A$. Select an edge $(\mathbf{x}, i) \in \nu(g) \setminus \nu(f)$. By 1, this determines $g(\mathbf{x})$ in terms of $f(\mathbf{x})$, so there is at most one such neighbour. □

The next lemma unveils a relation between the natural partition of a set $A \in \mathcal{N}_{\mathbf{ta}}$ and the map ψ_A. This relation will be important in understanding Lemma 4.3.

Lemma 4.2. *Let $A \in \mathcal{N}_{\mathbf{ta}}$ for some $\mathbf{a}$. Let $\mathcal{A}$ be the natural partition of A (recall that $\mathcal{A}$ contains the images of the discrete loops $\mathcal{F}$ of A) and let $a \in \mathcal{A}$. Then $\psi_A(a) \in \mathcal{A}$, and consequently ψ_A circularly permutes the elements of $\mathcal{A}$.*

Proof. Let $(\mathbf{x}, i), (\mathbf{y}, j) \in a$. We claim that $\psi_A(\mathbf{x}, i)$ and $\psi_A(\mathbf{y}, j)$ are in the same member of $\mathcal{A}$. For this we look at the loop structure $\lambda(A)$ of A and apply (6). Let $\alpha \in \lambda(A)$ be the continuous loop corresponding to a and write $\mathfrak{F}$ for the set $\cup_{\beta\in\lambda(A)} \operatorname{Im} \beta$. All loops in $\lambda(A)$ move up and left only. Suppose now that we start somewhere on the loop α and move up or right until we hit a loop in $\lambda(A)$. Conditional on not immediately hitting the loop we are already in, we will always hit the same loop in $\lambda(A)$, independently of the starting point on α that we chose. Therefore the points $\omega_{\mathfrak{F}}(\mu(\mathbf{x}, i), i)$ and $\omega_{\mathfrak{F}}(\mu(\mathbf{y}, j), j)$ lie in the same loop. Consequently (by (6)), the two edges $\psi_A(\mathbf{x}, i)$ and $\psi_A(\mathbf{y}, j)$ are elements of the member of $\mathcal{A}$ corresponding to that loop. This proves the claim. Therefore there exists a set $b \in \mathcal{A}$ such that $\psi_A(a) \subset b$. Since ψ_A is also a bijection, we must have $\psi_A(a) = b$, and ψ_A must in fact permute the elements of $\mathcal{A}$. To see that the permutation is circular, we observe that every element in $\mathcal{A}$ intersects $L_{(\mathbf{0},1)}$ (since $\mathcal{A} \subset \mathcal{N}_{\mathbf{t}(\mathbf{a}/\gcd \mathbf{a})}$), and that ψ_A circularly permutes the elements of $A \cap L_{(\mathbf{0},1)}$. □

Lemma 4.3. *Suppose that $A, B \in \mathcal{N}_{\mathbf{tn}}$ are disjoint and closed under the map $\pi_{12}^{A\cup B}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the natural partitions of A and B respectively. Then $\mathcal{C} := \mathcal{A} \cup \mathcal{B}$ is the natural partition of $C := A \cup B$. Moreover A and B are neighbours if and only if $\psi_C(\mathcal{A}) = \mathcal{B}$.*

Proof. First, if A and B are disjoint then $C \in \mathcal{N}_{\mathbf{t}(2\mathbf{n})}$, so the map π_{12}^C is well-defined. Since $\mathcal{C}$ is the finest partition of C such that its members are closed under the map

π_{12}^C, it must be a refinement of the partition $\{A, B\}$ of C. In fact, it is straightforward to deduce that $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$. By the previous lemma the map ψ_C permutes the elements of $\mathcal{C}$. Suppose first that $\psi_C(\mathcal{A}) \neq \mathcal{B}$, and assume that there exist neighbours f and g with $\nu(f) = A$ and $\nu(g) = B$. By assumption there exists an element $(\mathbf{x}, i) \in A$ such that $\psi_C(\mathbf{x}, i) \in A$. By Lemma 4.1 we have $f(\mathbf{x}) = g(\mathbf{x}) + 1$ (since $\phi_C(\mathbf{x}, i) \in A \setminus B$) and $f(\mathbf{x} + e_i) = g(\mathbf{x} + e_i) + 1$ (since $\phi_C(\mathbf{x} + e_i, i) = \psi_A(\mathbf{x}, i) \in A \setminus B$). This contradicts that $(\mathbf{x}, i)$ is a down step for f and not for g, hence such neighbours f and g cannot exist. Suppose now that $\psi_C(\mathcal{A}) = \mathcal{B}$, i.e., $\psi_C(A) = B$. Pick f such that $\nu(f) = A$. Define $g : V_{\mathbf{t}} \to \mathbb{R}$ by

$$g(\mathbf{x}) := f(\mathbf{x}) + 1 - 2 \cdot 1_{\phi_C(\mathbf{x},1) \in A} = f(\mathbf{x}) + 1 - 2 \cdot 1_{\phi_C(\mathbf{x},2) \in A}.$$

To see that this is well-defined we need to prove the equality. It suffices to show that $\phi_C(\mathbf{x}, 1)$ and $\phi_C(\mathbf{x}, 2)$ are in the same member of $\mathcal{C}$, since A is a union of members of $\mathcal{C}$. We now use (5) and reason as in the proof of Lemma 4.2. The elements $\omega_{\mathfrak{F}}(\mathbf{x}, 1)$ and $\omega_{\mathfrak{F}}(\mathbf{x}, 2)$ are in the image of one loop $\gamma \in \lambda(C)$, so both $\phi_C(\mathbf{x}, 1)$ and $\phi_C(\mathbf{x}, 2)$ are in the member of $\mathcal{C}$ that corresponds to this loop γ. This proves well-definedness. As $\psi_C(A) = B$ and $\psi_C(B) = A$, we have

$$1_{\phi_C(\mathbf{x}+e_i,i) \in A} - 1_{\phi_C(\mathbf{x},i) \in A} = 1_{(\mathbf{x},i) \in B} - 1_{(\mathbf{x},i) \in A}.$$

From this we deduce that

$$\begin{aligned} g(\mathbf{x} + e_i) - g(\mathbf{x}) &= f(\mathbf{x} + e_i) - f(\mathbf{x}) - 2\left(1_{\phi_C(\mathbf{x}+e_i,i) \in A} - 1_{\phi_C(\mathbf{x},i) \in A}\right) \\ &= -\mathbf{q}_i + 1 - 2 \cdot 1_{(\mathbf{x},i) \in B}. \end{aligned}$$

Therefore g is a height function with $\nu(g) = B$, and $g \sim f$ by construction. □

The previous lemma tells us that, conditional on A and B being disjoint and closed under the map $\pi_{12}^{A \cup B}$, the statement "$A \sim B$" depends only on the circular ordering $\psi_{A \cup B}$ of the $2 \gcd \mathbf{n}$ elements of $\mathcal{A} \cup \mathcal{B}$. If the circular ording $\psi_{A \cup B}$ of $\mathcal{A} \cup \mathcal{B}$ is such that $\psi_{A \cup B}(\mathcal{A}) = \mathcal{B}$ then we say that $\psi_{A \cup B}$ intertwines $\mathcal{A}$ and $\mathcal{B}$.

Lemma 4.4. *Let f and g be neighbours such that $\nu(f)$ and $\nu(g)$ are disjoint. Let $\mathfrak{F} = \cup_{\alpha \in \lambda(f)} \operatorname{Im} \alpha$ and $\mathfrak{G} = \cup_{\beta \in \lambda(g)} \operatorname{Im} \beta$. Then $\hat{g} - \hat{f}$ equals*

$$|V_{\mathbf{t}}|^{-1} \left(\operatorname{Vol}(\{\tilde{\mathbf{x}} \in C_{\mathbf{t}} : \omega_{\mathfrak{F} \cup \mathfrak{G}}(\tilde{\mathbf{x}}, 1) \in \mathfrak{G}\}) - \operatorname{Vol}(\{\tilde{\mathbf{x}} \in C_{\mathbf{t}} : \omega_{\mathfrak{F} \cup \mathfrak{G}}(\tilde{\mathbf{x}}, 1) \in \mathfrak{F}\}) \right).$$

In Figure 4a we have marked the two surfaces appearing in the lemma.

Proof. The set $C_{\mathbf{t}}$ is the disjoint union of $|V_{\mathbf{t}}|$ patches of the form $\{\mathbf{x} + s e_1 + s' e_2 : s, s' \in (-\frac{1}{2}, \frac{1}{2})\}$, where $\mathbf{x}$ ranges over $V_{\mathbf{t}}$, and a set of zero volume. It is clear that the volume of each of these patches is one. By Lemma 4.1,

$$g(\mathbf{x}) - f(\mathbf{x}) = 1_{\phi_{\nu(f) \cup \nu(g)}(\mathbf{x},1) \in \nu(g)} - 1_{\phi_{\nu(f) \cup \nu(g)}(\mathbf{x},1) \in \nu(f)}.$$

Let $\mathbf{x} \in V_\mathbf{t}$ and $\tilde{\mathbf{x}} \in \{\mathbf{x} + se_1 + s'e_2 : s, s' \in (-\frac{1}{2}, \frac{1}{2})\}$. We claim that $\phi_{\nu(f)\cup\nu(g)}(\mathbf{x}, 1) \in \nu(f) \setminus \nu(g)$ if and only if $\omega_{\mathfrak{F}\cup\mathfrak{G}}(\tilde{\mathbf{x}}, 1) \in \mathfrak{F} \setminus \mathfrak{G}$. Suppose that $\phi_{\nu(f)\cup\nu(g)}(\mathbf{x}, 1) \in \nu(f) \setminus \nu(g)$ and let $t = \min\{s \geq 0 : \mathbf{x} + se_1 \in \mathfrak{F} \cup \mathfrak{G}\}$. Then the following three statements follow from the construction of the loops in $\lambda(f)$, $\lambda(g)$ and $\mathcal{K}_{\mathbf{td}}$.

1. No loop in $\mathcal{K}_{\mathbf{td}}$ hits the set $\{\mathbf{x} + se_1 + s'e_2 : s, s' \in (-1/2, 1/2)\}$,
2. No loop in $\lambda(f)$ or $\lambda(g)$ hits $\{\mathbf{x} + se_1 + s'e_2 : s \in (-1/2, t), s' \in (-1/2, 1/2)\}$,
3. The set $\{\mathbf{x} + te_1 + se_2 : s \in (-1/2, 1/2)\}$ is contained in $\mathfrak{F} \setminus \mathfrak{G}$.

Hence $\omega_{\mathfrak{F}\cup\mathfrak{G}}(\tilde{\mathbf{x}}, 1) \in \mathfrak{F} \setminus \mathfrak{G}$. This proves our claim, where the other direction follows by interchanging f and g and the objects that derive from it. By the decomposition of $C_\mathbf{t}$ and the claim the volume difference in the statement of the lemma equals

$$\sum_{\mathbf{x}\in V_\mathbf{t}} \left(1_{\phi_{\nu(f)\cup\nu(g)}(\mathbf{x},1)\in\nu(g)} - 1_{\phi_{\nu(f)\cup\nu(g)}(\mathbf{x},1)\in\nu(f)}\right) = \sum_{\mathbf{x}\in V_\mathbf{t}} \left(g(\mathbf{x}) - f(\mathbf{x})\right) = |V_\mathbf{t}|(\hat{g} - \hat{f}).$$

This is the desired result. □

5. Embedding of paths in narrow strips

In the previous sections we have seen how to relate a shape to a set of loops in $\mathcal{K}_{\mathbf{td}}$. In this section we take a probabilistic perspective: what does a loop that is selected uniformly at random from $\mathcal{K}_{\mathbf{td}}$ look like? The main result of this section is Lemma 5.2, which tells us that a randomly selected loop is likely to be close to a straight diagonal loop if $\mathbf{t}$ is sufficiently large. To formalise the idea of being close to a diagonal loop, we introduce the notion of strips. For $R \subset \mathbb{R}$ define

$$U_{\mathbf{td},R} := \left\{\hat{\mathbf{0}} + se_1 + s'(-\mathbf{d}_2\mathbf{t}_1 e_1 + \mathbf{d}_1\mathbf{t}_2 e_2) : s \in R, s' \in \mathbb{R}\right\} \subset C_\mathbf{t}.$$

If $R = \{h\}$ for some h then $U_{\mathbf{td},R}$ is a diagonal loop. The winding numbers of $U_{\mathbf{td},\{h\}}$ are the same as the winding numbers of the loops in $\mathcal{K}_{\mathbf{td}}$. Note also that $U_{\mathbf{td},\{h\}}$ intersects the line $L_{(\mathbf{0},1)}$ exactly $\mathbf{d}_1$ times. If $R = [h - r/2, h + r/2]$ for some h and $r \in (0, \mathbf{t}_1/\mathbf{d}_1)$ then the set $U_{\mathbf{td},R}$ is a closed diagonal strip. If $r \geq \mathbf{t}_1/\mathbf{d}_1$ then $U_{\mathbf{td},[h-r/2,h+r/2]} = C_\mathbf{t}$. For fixed $\alpha \in \mathcal{K}_{\mathbf{td}}$, the collection of strips $U_{\mathbf{td},R}$ that contain α contains a unique smallest element. We prove this in the following lemma.

Lemma 5.1. *Let $\alpha \in \mathcal{K}_{\mathbf{td}}$. Then there exist numbers $h \in [0, \mathbf{t}_1/\mathbf{d}_1)$ and $r \in [0, \mathbf{t}_1/\mathbf{d}_1]$ such that*

$$U_{\mathbf{td},[h-r/2,h+r/2]} = \cap_{R:\mathrm{Im}\,\alpha\subset U_{\mathbf{td},R}} U_{\mathbf{td},R}.$$

Moreover if this intersection does not equal $C_\mathbf{t}$, then the numbers h and r are unique.

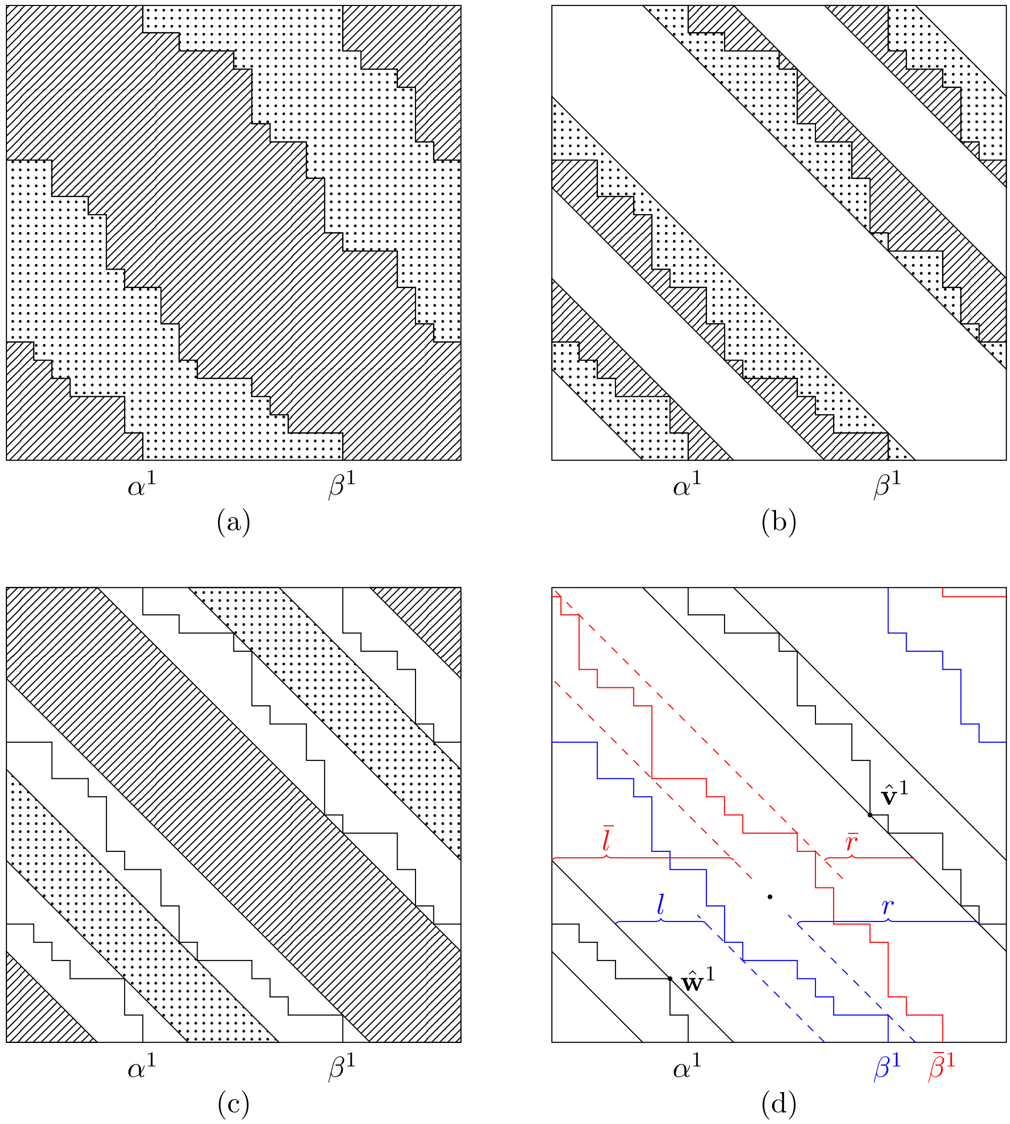

Figure 4: Two neighbours f and g with $\nu(f)$ and $\nu(g)$ disjoint. Write $\lambda(f) = \{\alpha^1\}$ and $\lambda(g) = \{\beta^1\}$. By Lemma 4.4, $|V_{\mathbf{t}}|(\hat{g} - \hat{f})$ equals the volume of the dotted area minus the volume of the striped area in (a). Figure (b) is related to the definition of κ. From the definition of κ it follows that $|V_{\mathbf{t}}|(\kappa(g) - \kappa(f))$ equals the difference in volume of the marked areas in (b). To calculate $|V_{\mathbf{t}}|(\hat{g} + \kappa(g) - \hat{f} - \kappa(f))$ we observe that the marked surfaces in (a) and (b) "cancel out" on the minimal strips of α^1 and β^1. Therefore this number equals the area difference in (c). This is Lemma 6.2. Figure (d) serves to illustrate the map τ from the proof of Lemma 6.3. The map τ interchanges the volumes of the areas left and right of the loop β^1. This is because $\bar{l} = r$ and $\bar{r} = l$.

Proof. Because $\operatorname{Im}\alpha$ is closed and connected, there must exist a closed connected set $R \subset \mathbb{R}$ such that $U_{\mathbf{td},R}$ equals the intersection in the statement of the lemma. If $U_{\mathbf{td},R} = C_{\mathbf{t}}$ then $U_{\mathbf{td},R} = U_{\mathbf{td},[-\mathbf{t}_1/(2\mathbf{d}_1),\mathbf{t}_1/(2\mathbf{d}_1)]}$, so we choose $h = 0$ and $r = \mathbf{t}_1/\mathbf{d}_1$. If $U_{\mathbf{td},R} \neq C_{\mathbf{t}}$ then the length of the interval R must be smaller than $\mathbf{t}_1/\mathbf{d}_1$, and we shift R an integral multiple of $\mathbf{t}_1/\mathbf{d}_1$ such that its midpoint is in $[0, \mathbf{t}_1/\mathbf{d}_1)$. Then there exist unique numbers $h \in [0, \mathbf{t}_1/\mathbf{d}_1)$ and $r \in [0, \mathbf{t}_1/\mathbf{d}_1)$ such that $R = [h - r/2, h + r/2]$. □

Write $h(\alpha)$ and $r(\alpha)$ for the unique numbers from the previous lemma, and $U_{\mathbf{td}}(\alpha)$ for the intersection from the same lemma (so $U_{\mathbf{td}}(\alpha) = U_{\mathbf{td},[h(\alpha)-r(\alpha)/2,h(\alpha)+r(\alpha)/2]}$). Call $U_{\mathbf{td}}(\alpha)$ the the minimal strip of α. By minimality, α intersects both $U_{\mathbf{td},\{h(\alpha)-r(\alpha)/2\}}$ and $U_{\mathbf{td},\{h(\alpha)+r(\alpha)/2\}}$, the two boundary lines of the minimal strip of α. This is illustrated in Figure 4d; the loop α^1 intersects the boundary lines of its minimal strip at the points $\hat{\mathbf{v}}^1$ and $\hat{\mathbf{w}}^1$. Now $r(\alpha)$ measures the extent to which the loop α remains close to a straight line, allowing us to state the main result of this section.

Lemma 5.2. *Let $\tilde{\mathbb{P}}_{\mathbf{pn}}$ be the uniform probability measure on $\mathcal{K}_{\mathbf{td}}$, so that the previously defined map $r : \mathcal{K}_{\mathbf{td}} \to [0, \mathbf{t}_1/\mathbf{d}_1]$ is a random variable on $\mathcal{K}_{\mathbf{td}}$. Then for all $\varepsilon > 0$ we have $\tilde{\mathbb{P}}_{\mathbf{pn}}((\mathbf{d}_1/\mathbf{t}_1)r \leq \varepsilon) \to 1$ as $\mathbf{p} \to \infty$.*

In order to prove this lemma, we first prove two auxiliary lemmas. We start by defining a suitable n-to-1 map (where $n = \mathbf{d}_2\mathbf{t}_1 + \mathbf{d}_1\mathbf{t}_2$) from a set with a nice combinatorial structure to $\mathcal{K}_{\mathbf{td}}$. Write $W_{x,y}$ for the set of walks on the integers from 0 to $x - y$ of length $x + y$, which is a finite set. So if $w \in W_{x,y}$ then w moves up by one exactly x times and down by one exactly y times. Write M for the matrix

$$\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Fix a pair $(\hat{\mathbf{x}}, w) \in \hat{V}_{\mathbf{t}} \times W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1}$. Write $\tilde{w} = (\tilde{w}_t)_{t\in[0,n]} \subset \mathbb{R}^2$ for the linear interpolation of $(k, w_k)_{0\leq k\leq n} \subset \mathbb{Z}^2$. Then $(\tilde{w}_t M)_{t\in[0,n]}$ is a path in $\mathbb{R}^2$ and $(\hat{\mathbf{x}} + \tilde{w}_t M)_{t\in[0,n]}$ is a path in $C_{\mathbf{t}}$.

Lemma 5.3. *The map*

$$\zeta : \hat{V}_{\mathbf{t}} \times W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1} \to \mathcal{K}_{\mathbf{td}},\ (\hat{\mathbf{x}}, w) \mapsto (\hat{\mathbf{x}} + \tilde{w}_t M)_{t\in[0,n]}$$

is well-defined and n-to-1.

Proof. Fix $(\hat{\mathbf{x}}, w) \in \hat{V}_{\mathbf{t}} \times W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1}$. To show well-definedness it suffices to demonstrate that the restriction of $\zeta(\hat{\mathbf{x}}, w)$ to integral times is a walk in $\hat{V}_{\mathbf{t}}$ that satisfies Statement 3 from Lemma 3.1. If $w_{k+1} = w_k + 1$ then $\zeta(\hat{\mathbf{x}}, w)_{k+1} = \zeta(\hat{\mathbf{x}}, w)_k + e_2$ (since $(1, 1)M = (0, 1)$) and if $w_{k+1} = w_k - 1$ then $\zeta(\hat{\mathbf{x}}, w)_{k+1} = \zeta(\hat{\mathbf{x}}, w)_k - e_1$ (since $(1, -1)M = (-1, 0)$). Since w moves up $\mathbf{d}_1\mathbf{t}_2$ times and down $\mathbf{d}_2\mathbf{t}_1$ times, Statement 3 indeed holds for the loop $\zeta(\hat{\mathbf{x}}, w)$. Note that $\zeta(\hat{\mathbf{x}}, w)_0 = \hat{\mathbf{x}} \in \hat{V}_{\mathbf{t}}$, and inductively $\zeta(\hat{\mathbf{x}}, w)_k \in \hat{V}_{\mathbf{t}}$ for any integral k. This proves well-definedness. The map ζ would have been a bijection, had we not identified loops in $\mathcal{K}_{\mathbf{td}}$ as we did (in the end of Section 3). If we pick $\alpha \in \mathcal{K}_{\mathbf{td}}$ and a starting point for this loop, then it is straightforward to construct the unique element $(\hat{\mathbf{x}}, w)$ that is mapped to α, such that also the starting points of the loops match. We recall from the end of Section 3 that each equivalence class from the identification contains n elements. Therefore ζ is n-to-1. □

The uniform probability measure on $W_{x,y}$ is understood relatively well, so we prefer to measure "closeness to a line" in that space first. Define

$$W_{x,y}(b) := \left\{ w \in W_{x,y} : \left| w_k - k\tfrac{x-y}{x+y} \right| \leq b \text{ for all } 0 \leq k \leq x+y \right\}.$$

If $w \in W_{x,y}(b)$ for b relatively small then $(k, w_k)_{0 \leq k \leq n}$ remains close to a diagonal line.

Lemma 5.4. *Let $\hat{\mathbb{P}}_{x,y}$ denote the uniform probability measure on $W_{x,y}$. Then for all $\varepsilon > 0$ we have*

$$\lim_{x,y\to\infty} \hat{\mathbb{P}}_{x,y} \left(W_{x,y} \left(\tfrac{xy}{x+y} \varepsilon \right) \right) = 1, \tag{7}$$

regardless of the relative speed at which x and y approach infinity.

Proof. By symmetry arguments, the probability in (7) is invariant under interchanging x and y. We may therefore assume, without loss of generality, that $x \geq y$. Note that $W_{x,y}(b') \subset W_{x,y}(b)$ whenever $b' \leq b$, and since $\frac{1}{2}y\varepsilon = \frac{xy}{2x}\varepsilon \leq \frac{xy}{x+y}\varepsilon$, it suffices to demonstrate that

$$\lim_{x,y\to\infty} \hat{\mathbb{P}}_{x,y} \left(W_{x,y} \left(4y\varepsilon \right) \right) = 1,$$

where we have replaced our original ε by a smaller one. Define

$$R_{x,y,k}(b) := \{ |w_k - k\tfrac{x-y}{x+y}| \leq b \} = \{ |\tfrac{1}{2}(w_k + k) - k\tfrac{x}{x+y}| \leq \tfrac{1}{2}b \} \subset W_{x,y}. \tag{8}$$

It is nontrivial but straightforward to check that

$$\cap_{0 \leq k < 1/\varepsilon} R_{x,y,\lceil k(x+y)\varepsilon \rceil}(2y\varepsilon) \subset W_{x,y}(4y\varepsilon),$$

so if we force the walk to pass through $\lceil 1/\varepsilon \rceil$ narrow gates at $\lceil 1/\varepsilon \rceil$ well-chosen times then the walk is guaranteed to stay close to the diagonal at all times. It now suffices to show that

$$\lim_{x,y\to\infty} \inf_{0 \leq k \leq x+y} \hat{\mathbb{P}}_{x,y} \left(R_{x,y,k}(2y\varepsilon) \right) = 1.$$

Now observe (8), and note that the distribution of $\frac{1}{2}(w_k + k)$ in $\hat{\mathbb{P}}_{x,y}$ is the hypergeometric distribution $\text{Hyper}(x+y, x, k)$. This distribution has mean $\frac{x}{x+y}$ and variance $\frac{xyk(x+y-k)}{(x+y)^2(x+y-1)}$. By Chebyshev's inequality we conclude that

$$\sup_{0 \leq k \leq x+y} \hat{\mathbb{P}}_{x,y} \left(\left| \tfrac{1}{2}(w_k + k) - k\tfrac{x}{x+y} \right| > y\varepsilon \right) \leq \sup_{0 \leq k \leq x+y} \frac{\text{Var}\left(\frac{1}{2}(w_k+k)\right)}{(y\varepsilon)^2} \to_{x,y\to\infty} 0.$$

This finishes the proof of the lemma. □

Finally we use this lemma and the map ζ to prove Lemma 5.2.

Proof of Lemma 5.2. Let $\hat{\mathbb{P}}_{\mathbf{pn}}$ denote the uniform probability measure on the finite set $\hat{V}_{\mathbf{t}} \times W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1}$. Now $\hat{\mathbf{x}}$, w and ζ are random variables, taking values in the spaces $\hat{V}_{\mathbf{t}}$, $W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1}$ and $\mathcal{K}_{\mathbf{td}}$ respectively. Since the map ζ is n-to-1, it is (as a random variable) uniformly distributed in $\mathcal{K}_{\mathbf{td}}$. By the previous lemma, the probability of the event

$$w \in W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1}(\mathbf{t}_1\mathbf{t}_2\varepsilon/n) = W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1}\left(\frac{\mathbf{d}_1\mathbf{t}_2\mathbf{d}_2\mathbf{t}_1}{\mathbf{d}_1\mathbf{t}_2+\mathbf{d}_2\mathbf{t}_1}\frac{\varepsilon}{\mathbf{d}_1\mathbf{d}_2}\right)$$

goes to one as $\mathbf{p} \to \infty$ (note that $\mathbf{d}_1\mathbf{t}_2, \mathbf{d}_2\mathbf{t}_1 \to \infty$ as $\mathbf{p}_1, \mathbf{p}_2 \to \infty$). Therefore it suffices to prove that $(\mathbf{d}_1/\mathbf{t}_1)r(\zeta) \leq \varepsilon$ whenever w is contained in this set. Pick $(\hat{\mathbf{x}}, w) \in \hat{V}_{\mathbf{t}} \times W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1}$ such that w is contained in $W_{\mathbf{d}_1\mathbf{t}_2,\mathbf{d}_2\mathbf{t}_1}(\mathbf{t}_1\mathbf{t}_2\varepsilon/n)$, and write $q = (\mathbf{d}_1\mathbf{t}_2 - \mathbf{d}_2\mathbf{t}_1)/n$. Then

$$(k, w_k)_{0\leq k\leq n} \subset \left\{s(0,1) + s'(1,q) \in \mathbb{R}^2 : s \in [-\mathbf{t}_1\mathbf{t}_2\varepsilon/n, \mathbf{t}_1\mathbf{t}_2\varepsilon/n], s' \in \mathbb{R}\right\} =: \tilde{U}.$$

Since $\tilde{U}$ is convex, the path $\tilde{w}$, the linear interpolation of $(k, w_k)_{0\leq k\leq n}$, is also contained in $\tilde{U}$. It follows that $\zeta = (\hat{\mathbf{x}} + \tilde{w}_t M)_{t\in[0,n]} \subset \hat{\mathbf{x}} + \tilde{U}M$. It suffices to show that $\hat{\mathbf{x}} + \tilde{U}M$ fits in a strip of width $r \leq (\mathbf{t}_1/\mathbf{d}_1)\varepsilon$. We assume, without loss of generality, that $\hat{\mathbf{x}} = \hat{\mathbf{0}}$. Observe that

$$(0,1)M = (e_1 + e_2)/2 \quad \text{and} \quad (1,q)M = (-\mathbf{d}_2\mathbf{t}_1 e_1 + \mathbf{d}_1\mathbf{t}_2 e_2)/n.$$

Now

$$\begin{aligned}
\zeta(\hat{\mathbf{x}}, w) &\subset \hat{\mathbf{0}} + \tilde{U}M \\
&= \hat{\mathbf{0}} + \{s(e_1+e_2)/2 + s'(-\mathbf{d}_2\mathbf{t}_1 e_1 + \mathbf{d}_1\mathbf{t}_2 e_2) : s \in [-\mathbf{t}_1\mathbf{t}_2\varepsilon/n, \mathbf{t}_1\mathbf{t}_2\varepsilon/n], s' \in \mathbb{R}\} \\
&= \hat{\mathbf{0}} + \{sne_1/(2\mathbf{d}_1\mathbf{t}_2) + s'(-\mathbf{d}_2\mathbf{t}_1 e_1 + \mathbf{d}_1\mathbf{t}_2 e_2) : s \in [-\mathbf{t}_1\mathbf{t}_2\varepsilon/n, \mathbf{t}_1\mathbf{t}_2\varepsilon/n], s' \in \mathbb{R}\} \\
&= \hat{\mathbf{0}} + \{se_1 + s'(-\mathbf{d}_2\mathbf{t}_1 e_1 + \mathbf{d}_1\mathbf{t}_2 e_2) : s \in [-(\mathbf{t}_1/\mathbf{d}_1)\varepsilon/2, (\mathbf{t}_1/\mathbf{d}_1)\varepsilon/2], s' \in \mathbb{R}\} \\
&= U_{\mathbf{td},[-(\mathbf{t}_1/\mathbf{d}_1)\varepsilon/2,(\mathbf{t}_1/\mathbf{d}_1)\varepsilon/2]}
\end{aligned}$$

Hence $r(\zeta) = r(\zeta(\hat{\mathbf{x}}, w)) \leq (\mathbf{t}_1/\mathbf{d}_1)\varepsilon$. □

We now understand the asymptotic behavior of the random variable $(\mathbf{d}_1/\mathbf{t}_1)r(\alpha)$, the normalised width of the minimal strip of a randomly chosen loop in $\mathcal{K}_{\mathbf{td}}$. The following lemma addresses the distribution of the random variable $h(\alpha)$, the "reference point" of the minimal strip of a randomly chosen loop.

Lemma 5.5. *Let $\tilde{\mathbb{P}}_{\mathbf{pn}}$ be the uniform probability measure on $\mathcal{K}_{\mathbf{td}}$, so that the previously defined map h is a random variable taking values in $[0, \mathbf{t}_1/\mathbf{d}_1)$. Then the distribution of the random variable $(\mathbf{d}_1/\mathbf{t}_1)h(\alpha)$ in $(\mathcal{K}_{\mathbf{td}}, \tilde{\mathbb{P}}_{\mathbf{pn}})$ converges to the continuous uniform distribution on the unit interval as $\mathbf{p} \to \infty$.*

Proof. Write α for a generic element in the sample space $\mathcal{K}_{\mathbf{td}}$, so that $h = h(\alpha)$ and $r = r(\alpha)$. The random variable $(\mathbf{d}_1/\mathbf{t}_1)h(\alpha)$ takes values in the set $[0, 1)$. Conditional on $r(\alpha) < \mathbf{t}_1/\mathbf{d}_1$, the distribution of $(\mathbf{d}_1/\mathbf{t}_1)h(\alpha)$ is invariant under translation by integral multiples of $\mathbf{d}_1/\mathbf{t}_1$ (modulo 1). This is because the map

$$\mathcal{K}_{\mathbf{td}} \to \mathcal{K}_{\mathbf{td}},\ \alpha \mapsto \alpha + e_1$$

is a bijection, with, conditional on $r(\alpha) < \mathbf{t}_1/\mathbf{d}_1$, $(\mathbf{d}_1/\mathbf{t}_1)h(\alpha + e_1) = (\mathbf{d}_1/\mathbf{t}_1)h(\alpha) + (\mathbf{d}_1/\mathbf{t}_1)$ (modulo 1). By Lemma 5.2 the probability of the event $\{r(\alpha^1) < \mathbf{t}_1/\mathbf{d}_1\}$ goes to one as $\mathbf{p} \to \infty$. The result follows since $(\mathbf{d}_1/\mathbf{t}_1) \to 0$ as $\mathbf{p}_1 \to \infty$. □

We now prove some results that we need later.

Lemma 5.6. *If $\alpha \in \mathcal{K}_{\mathbf{td}}$ and $r(\alpha) < \mathbf{t}_1/\mathbf{d}_1$ then α is a simple loop.*

Proof. The set $U_{\mathbf{td}}(\alpha)$ is a proper subset of $C_{\mathbf{t}}$ if and only if $r(\alpha) < \mathbf{t}_1/\mathbf{d}_1$. Therefore it is, under the hypothesis of the lemma, homeomorphic to a cylinder. Recall that $\dot{\alpha}_t \in \{-e_1, e_2\}$ for any nonintegral t. If α is not simple then α must wind around the cylinder more than once. But then α winds around the torus more than $\mathbf{d}_2$ times in the horizontal direction and more than $\mathbf{d}_1$ times in the vertical direction. This contradicts that $\alpha \in \mathcal{K}_{\mathbf{td}}$, hence α must be simple. □

In the context of our model we need to sample $2 \gcd \mathbf{n}$ loops, $\gcd \mathbf{n}$ loops for each shape of a pair of neighbours. From now on we write $\tilde{\mathbb{P}}_{\mathbf{pn}}$ for the uniform probability measure on $\mathcal{K}_{\mathbf{td}}^{2\gcd \mathbf{n}} = \mathcal{K}_{\mathbf{td}}^{\gcd \mathbf{n}} \times \mathcal{K}_{\mathbf{td}}^{\gcd \mathbf{n}}$, and write $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha^1, ..., \alpha^{\gcd \mathbf{n}}, \beta^1, ...\beta^{\gcd \mathbf{n}})$ for a generic element in this sample space. We define the event

$$\mathcal{D}_{\mathbf{tn}} := \{\text{the } 2 \gcd \mathbf{n} \text{ minimal strips of } (\boldsymbol{\alpha}, \boldsymbol{\beta}) \text{ are pairwise disjoint}\} \subset \mathcal{K}_{\mathbf{td}}^{2\gcd \mathbf{n}}.$$

Lemma 5.7. *Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{D}_{\mathbf{tn}}$. Then there exist unique shapes $A, B \in \mathcal{S}_{\mathbf{pn}}$ such that $\lambda(A) = \{\alpha^1, ..., \alpha^{\gcd \mathbf{n}}\}$ and $\lambda(B) = \{\beta^1, ..., \beta^{\gcd \mathbf{n}}\}$. Moreover A and B are disjoint and closed under the map $\pi_{12}^{A \cup B}$.*

Proof. The $2 \gcd \mathbf{n}$ loops lie in disjoint strips, and are therefore disjoint. By the previous lemma the loops are also simple. As in the proof of Lemma 3.2, we have $A = \mu^{-1}(\cup_i \operatorname{Im} \alpha^i)$ and $B = \mu^{-1}(\cup_i \operatorname{Im} \beta^i)$, and those sets are in $\mathcal{N}_{\mathbf{tn}}$. Since the loops are disjoint, A and B must be disjoint. Write $\mathcal{A}$ and $\mathcal{B}$ for the natural partitions of A and B respectively. We apply the same lemma to see that $C = A \cup B = \mu^{-1}(\cup_i (\operatorname{Im} \alpha^i \cup \operatorname{Im} \beta^i)) \in \mathcal{N}_{\mathbf{t(2n)}}$, and $\lambda(C) = \{\alpha^1, ..., \alpha^{\gcd \mathbf{n}}, \beta^1, ..., \beta^{\gcd \mathbf{n}}\}$. Therefore the natural partition of C is $\mathcal{A} \cup \mathcal{B}$. Hence A is a union of subsets of C that are closed under the map π_{12}^{C}, so A must be closed under the map $\pi_{12}^{A \cup B}$. The same holds for B. □

Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{D}_{\mathbf{tn}}$ and let A and B be the sets in $\mathcal{N}_{\mathbf{tn}}$ corresponding to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively. Lemma 4.3 tells us that A and B are neighbours if the circular ordering $\psi_{A \cup B}$ of $\mathcal{A}$ and $\mathcal{B}$ intertwines $\mathcal{A}$ and $\mathcal{B}$. Each member $a^i \in \mathcal{A}$ is associated with a loop α^i, and each $b^i \in \mathcal{B}$ is associated with a loop β^i. Therefore the circular ordering $\psi_{A\cup B}$ of $\mathcal{A}\cup\mathcal{B}$ extends to a circular ordering of $\{\alpha^1, ..., \alpha^{\gcd \mathbf{n}}, \beta^1, ..., \beta^{\gcd \mathbf{n}}\}$. However, we can also circularly order these $2\gcd\mathbf{n}$ loops by inspecting the order "$>$" on the set $\{h(\alpha^1), ..., h(\alpha^{\gcd\mathbf{n}}), h(\beta^1), ..., h(\beta^{\gcd\mathbf{n}})\}$. Note that the strips are disjoint, so that the value of h must be different for every loop. It is straightforward to check that the two orderings coincide. This allows us to rewrite the condition in Lemma 4.3.

Lemma 5.8. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{D}_{\mathbf{tn}}$ and let $A, B \in \mathcal{N}_{\mathbf{tn}}$ be the sets corresponding to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively. Then A and B are neighbours if and only if the ordering "$>$" intertwines the sets $\{h(\alpha^1), ..., h(\alpha^{\gcd\mathbf{n}})\}$ and $\{h(\beta^1), ..., h(\beta^{\gcd\mathbf{n}})\}$, i.e., if (up to reordering the indices in both sets) either*

$$h(\alpha^1) < h(\beta^1) < ... < h(\alpha^{\gcd\mathbf{n}}) < h(\beta^{\gcd\mathbf{n}}) \tag{9}$$

or

$$h(\beta^1) < h(\alpha^1) < ... < h(\beta^{\gcd\mathbf{n}}) < h(\alpha^{\gcd\mathbf{n}}).$$

Finally, the probability of the event $\mathcal{D}_{\mathbf{tn}}$ becomes large as $\mathbf{p} \to \infty$.

Lemma 5.9. *We have* $\lim_{\mathbf{p}\to\infty} \tilde{\mathbb{P}}_{\mathbf{pn}}(\mathcal{D}_{\mathbf{tn}}) = 1$.

Proof. It suffices to prove that the probability of the event

$$\{\text{the loops } \alpha^1 \text{ and } \alpha^2 \text{ lie in disjoint strips}\} \subset \mathcal{K}_{\mathbf{td}}^{2\gcd\mathbf{n}}$$

goes to one as $\mathbf{p} \to \infty$. Fix $\varepsilon > 0$. By Lemmas 5.2 and 5.5, we may in the limit pretend that $r(\alpha^1), r(\alpha^2) \leq \varepsilon \mathbf{t}_1/\mathbf{d}_1$ and that $\frac{\mathbf{d}_1}{\mathbf{t}_1}(h(\alpha^1), h(\alpha^2))$ has the continuous uniform distribution on $[0,1]^2$. Therefore

$$\begin{aligned}
&\limsup_{\mathbf{p}\to\infty} \tilde{\mathbb{P}}_{\mathbf{pn}}(\text{the minimal strips of } \alpha^1 \text{ and } \alpha^2 \text{ intersect}) \\
&\leq \mathbb{P}\left(\text{the strip } U_{\mathbf{td},(\mathbf{t}_1/\mathbf{d}_1)[Y_1-\varepsilon/2, Y_1+\varepsilon/2]} \text{ intersects } U_{\mathbf{td},(\mathbf{t}_1/\mathbf{d}_1)[Y_2-\varepsilon/2, Y_2+\varepsilon/2]}\right) \\
&= \mathbb{P}(\text{the set } [Y_1 - \varepsilon/2, Y_1 + \varepsilon/2] + \mathbb{Z} \text{ intersects } [Y_2 - \varepsilon/2, Y_2 + \varepsilon/2] + \mathbb{Z}) \\
&= 2\varepsilon,
\end{aligned}$$

where (Y_1, Y_2) has the continuous uniform distribution on $[0,1]^2$ in some measure $\mathbb{P}$. This is the desired result as ε may be chosen arbitrarily small. □

6. Correction of the average height process

Recall that $X^{\mathbf{pn}}$ is a random walk on $\mathbf{pn}$-periodic height functions. The corresponding average height process $\hat{X}^{\mathbf{pn}}$ is almost an additive functional of the walk on shapes. If the shapes of $X_n^{\mathbf{pn}}$ and $X_{n+1}^{\mathbf{pn}}$ are not the same, then by Lemma 4.1, Statement 2, we can infer the difference in average height $\hat{X}_{n+1}^{\mathbf{pn}} - \hat{X}_n^{\mathbf{pn}}$ from the shapes $[X_n^{\mathbf{pn}}]$ and $[X_{n+1}^{\mathbf{pn}}]$ (we will not do this explicitly). If the shapes of $X_n^{\mathbf{pn}}$ and $X_{n+1}^{\mathbf{pn}}$ are the same then we do not know if $X^{\mathbf{pn}}_{n+1} = X^{\mathbf{pn}}_n + 1$ or $X^{\mathbf{pn}}_{n+1} = X^{\mathbf{pn}}_n - 1$, so $\hat{X}^{\mathbf{pn}}$ is not formally an additive functional of the walk on shapes. We first state a general theorem regarding additive functionals of Markov chains, and include a proof for completeness. If the distribution of some process H (such as an additive functional of a Markov chain) converges to that of a Brownian motion under diffusive scaling then we write $\sigma^2(H)$ for the diffusivity of the limit distribution. Call $\sigma^2(H)$ the diffusivity of the process H.

Theorem 6.1. *Let S be a finite set, $d : S \times S \to \mathbb{R}$ an antisymmetric map and $(X_n)_{n\geq 0}$ an irreducible reversible Markov chain on S starting from its invariant distribution. Define the process H by $H_n := \sum_{k=0}^{n-1} d(X_k, X_{k+1})$. A map $\kappa^* \in \mathbb{R}^S$ minimises*

$$E(\kappa) := \mathbb{E}\left((H_1 + \kappa(X_1) - H_0 - \kappa(X_0))^2\right)$$

over $\kappa \in \mathbb{R}^S$ if and only if $(H_n + \kappa^*(X_n))_{n\geq 0}$ is a martingale. Such a map κ^* exists and is unique up to constant differences. Moreover the law of H converges to that of a Brownian motion of diffusivity $E(\kappa^*)$.

The minimiser κ^* is called the corrector of H.

Proof of Theorem 6.1. By writing the expectation as a finite sum over the entries of the transition matrix of X we see that the objective function is quadratic. Therefore it is convex and the set of minima is an affine subspace of $\mathbb{R}^S$. Adding a constant to the map κ does not change $E(\kappa)$. We note that

$$E(\kappa) \to \infty \quad \text{as} \quad ||\kappa|| \to \infty$$

if we keep $\kappa(s)$ fixed for some $s \in S$. Hence a minimiser of E must exist and is unique up to constant differences. Write κ^* for such a minimiser, so for all $s \in S$ we have

$$\frac{\partial}{\partial \kappa^*(s)} E(\kappa^*) = 0.$$

By moving the derivative into the expectation and using the detailed balance equations and antisymmetry of d it is straightforward to check that

$$\frac{\partial}{\partial \kappa(s)} E(\kappa) = -4\mathbb{P}(X_0 = s)\mathbb{E}\left(H_1 + \kappa(X_1) - H_0 - \kappa(X_0) | X_0 = s\right).$$

From this we conclude that

$$\mathbb{E}\left(H_1 + \kappa^*(X_1) - H_0 - \kappa^*(X_0) | X_0 = s\right) = 0 \quad \text{for all } s \in S,$$

and therefore $(H_n + \kappa^*(X_n))_{n\geq 0}$ must be a martingale. If $(H_n + \kappa^*(X_n))_{n\geq 0}$ is a martingale then by reversing the previous argument, κ^* is a local minimum of the objective function. The objective function is convex, so that κ^* must be a global minimum. Finally by standard arguments the distribution of H converges to that of a Brownian motion under diffusive scaling. If $(H_n + \kappa^*(X_n))_{n\geq 0}$ is a martingale then its increments are orthogonal and identically distributed, so that

$$\begin{aligned}\sigma^2(H) &= \lim_{n\to\infty} \mathbb{E}\left(\left(n^{-1/2}H_n\right)^2\right) \\ &= \lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\left(\sum_{k=0}^{n-1}\left(H_{k+1} + \kappa^*(X_{k+1}) - H_k - \kappa^*(X_k)\right)\right)^2\right) = E(\kappa^*).\end{aligned}$$

The second equality follows from the fact that $\sum_{k=0}^{n-1}\left(\kappa(X_{k+1}) - \kappa(X_k)\right) = \kappa(X_n) - \kappa(X_0)$ is bounded. The penultimate expression is constant over n by orthogonality of martingale increments, and setting $n = 1$ gives the final equality. This finishes the proof of the theorem. □

The goal of this section is to construct a process $Z^{\mathbf{pn}}$ meeting the following criteria:

1. The difference $Z^{\mathbf{pn}} - \hat{X}^{\mathbf{pn}}$ is small in the sense that $\lim_{\mathbf{p}\to\infty} \sigma^2(Z^{\mathbf{pn}} - \hat{X}^{\mathbf{pn}}) = 0$,
2. The corrector κ for the process $Z^{\mathbf{pn}}$ takes a simple form,
3. The random variable $Z_1^{\mathbf{pn}} + \kappa(X_1^{\mathbf{pn}}) - Z_0^{\mathbf{pn}} - \kappa(X_0^{\mathbf{pn}})$ takes a simple form.

From 1 it follows that $\lim_{\mathbf{p}} \sigma^2(\hat{X}^{\mathbf{pn}}) = \lim_{\mathbf{p}} \sigma^2(Z^{\mathbf{pn}})$, and 2 and 3 allow us to prove that $\lim_{\mathbf{p}} \sigma^2(Z^{\mathbf{pn}}) = (1 + 2\gcd \mathbf{n})^{-1}$. Let $\mathbb{P}_{\mathbf{pn}}$ be a measure for $X^{\mathbf{pn}}$ such that the distribution of $([X_n^{\mathbf{pn}}])_{n\geq 0}$, the walk on shapes, is invariant. All objects in this section depend on $\mathbf{p}$ and $\mathbf{n}$, but this dependence will not be made explicit unless necessary. In the final two sections we use the letters A and B for elements in $\mathcal{S}_{\mathbf{pn}}$, so A and B are equivalence classes of height functions.

Let κ be a map from $\mathcal{S}_{\mathbf{pn}}$ (the set of shapes) to $\mathbb{R}$, which is bounded as $\mathcal{S}_{\mathbf{pn}}$ is finite, and write also κ for the process $(\kappa([X_n]))_{n\geq 0}$. Define the antisymmetric map

$$y : \mathcal{S}_{\mathbf{pn}} \times \mathcal{S}_{\mathbf{pn}} \to \mathbb{R},\ (A, B) \mapsto \begin{cases} \hat{g} - \hat{f} & \text{if } A \sim B \text{ and } A \neq B \\ 0 & \text{otherwise} \end{cases},$$

where f and g are neighbours chosen from the equivalence classes A and B respectively. Lemma 4.1, Statement 2 guarantees that y is well-defined. To simplify notation we write $y(f, g)$ for $y([f], [g])$. Observe that

$$\hat{X}_n - \hat{X}_0 + \kappa_n - \kappa_0 = \sum_{k=0}^{n-1}\left(y(X_k, X_{k+1}) + 1_{X_{k+1}=X_k+1} - 1_{X_{k+1}=X_k-1} + \kappa_{k+1} - \kappa_k\right).$$

This process is a martingale if and only if κ is a corrector for the process Y defined by

$$Y_n = \sum_{k=0}^{n-1} y(X_k, X_{k+1}).$$

Choose κ to be the corrector for Y (and thus for $\hat{X}$), which exists by the theorem. Then

$$\begin{aligned}\sigma^2(\hat{X}) &= \mathbb{E}\left((\hat{X}_1 + \kappa_1 - \hat{X}_0 - \kappa_0)^2\right) \\ &= \mathbb{E}\left((y(X_0, X_1) + 1_{X_1 = X_0+1} - 1_{X_1 = X_0 - 1} + \kappa_1 - \kappa_0)^2\right) \\ &= \mathbb{E}\left((y(X_0, X_1) + \kappa_1 - \kappa_0)^2\right) + \mathbb{P}([X_1] = [X_0]) \\ &= \sigma^2(Y) + \mathbb{P}([X_1] = [X_0]). \end{aligned} \tag{10}$$

We shall have three antisymmetric maps and three additive functionals of $[X]$;

$$\begin{aligned} y &: \mathcal{S}_{\mathbf{pn}} \times \mathcal{S}_{\mathbf{pn}} \to \mathbb{R}, & Y_n &:= \textstyle\sum_{k=0}^{n-1} y(X_k, X_{k+1}), \\ z &: \mathcal{S}_{\mathbf{pn}} \times \mathcal{S}_{\mathbf{pn}} \to \mathbb{R}, & Z_n &:= \textstyle\sum_{k=0}^{n-1} z(X_k, X_{k+1}), \\ d &: \mathcal{S}_{\mathbf{pn}} \times \mathcal{S}_{\mathbf{pn}} \to \mathbb{R}, \text{ and} & D_n &:= \textstyle\sum_{k=0}^{n-1} d(X_k, X_{k+1}), \end{aligned}$$

satisfying $y = z + d$, and consequently $Y = Z + D$.

Define $\mathcal{P}_{\mathbf{pn}}$ to be the set of pairs $(A, B) \in \mathcal{S}_{\mathbf{pn}} \times \mathcal{S}_{\mathbf{pn}}$ such that the $2 \gcd \mathbf{n}$ minimal strips of the loops in $\lambda(A)$ and $\lambda(B)$ are pairwise disjoint. We shall see in Lemma 7.1 that for large $\mathbf{p}$, the pair $(X_0^{\mathbf{pn}}, X_1^{\mathbf{pn}})$ is very likely to be in $\mathcal{P}_{\mathbf{pn}}$. Define

$$z(A, B) := \begin{cases} y(A, B) & \text{if } (A, B) \in \mathcal{P}_{\mathbf{pn}} \\ \kappa(A) - \kappa(B) & \text{if } (A, B) \notin \mathcal{P}_{\mathbf{pn}} \end{cases}$$

and $d := y - z$, where the map $\kappa : \mathcal{S}_{\mathbf{pn}} \to \mathbb{R}$ is yet to be defined. Pick $A \in \mathcal{S}_{\mathbf{pn}}$; we will now define $\kappa(A)$. Let $\mathfrak{F} := \cup_{\alpha \in \lambda(A)} \operatorname{Im} \alpha$, let $\mathfrak{A} := \cup_{\alpha \in \lambda(A)} U_{\mathbf{td}}(\alpha)$ and write $\partial\mathfrak{A}$ for the topological boundary of $\mathfrak{A}$. Define $\kappa(A)$ by

$$|V_{\mathbf{t}}|\kappa(A) := \operatorname{Vol}(\{\tilde{\mathbf{x}} \in \mathfrak{A} : \omega_{\mathfrak{F} \cup \partial\mathfrak{A}}(\tilde{\mathbf{x}}, 1) \in \partial\mathfrak{A}\}) - \operatorname{Vol}(\{\tilde{\mathbf{x}} \in \mathfrak{A} : \omega_{\mathfrak{F} \cup \partial\mathfrak{A}}(\tilde{\mathbf{x}}, 1) \in \mathfrak{F}\}).$$

In Figure 4b the areas corresponding to the volumes have been marked. A bound on the map κ is given by $|\kappa(A)| \leq |V_{\mathbf{t}}|^{-1} \operatorname{Vol}(\mathfrak{A}) \leq 1$.

We have now defined the process $Z^{\mathbf{pn}}$ and its corrector. In the next lemma we simplify $Z_1^{\mathbf{pn}} + \kappa(X_1^{\mathbf{pn}}) - Z_0^{\mathbf{pn}} - \kappa(X_0^{\mathbf{pn}})$, and in Lemma 6.3 we prove that κ is indeed the corrector of $Z^{\mathbf{pn}}$. In Lemma 7.1 we prove that $\lim_{\mathbf{p} \to \infty} \sigma^2(Z^{\mathbf{pn}} - \hat{X}^{\mathbf{pn}}) = 0$.

Lemma 6.2. *Let $(A, B) \in \mathcal{P}_{\mathbf{pn}}$. Index the loops α^i in $\lambda(A)$ and the loops β^i in $\lambda(B)$ such that $h(\alpha^i) < h(\alpha^j)$ and $h(\beta^i) < h(\beta^j)$ for $i < j$. Then*

$$z(A, B) + \kappa(B) - \kappa(A) = 2(\mathbf{d}_1/\mathbf{t}_1)\left(\textstyle\sum_i h(\beta^i) - h(\alpha^i)\right) - 1_{h(\alpha^1) < h(\beta^1)} + 1_{h(\beta^1) < h(\alpha^1)}. \tag{11}$$

Moreover, if $(A, B) \notin \mathcal{P}_{\mathbf{pn}}$, then

$$z(A, B) + \kappa(B) - \kappa(A) = 0. \tag{12}$$

Proof. If $(A,B) \notin \mathcal{P}_{\mathbf{pn}}$ then (12) follows immediately from the definition of z. Let $(A,B) \in \mathcal{P}_{\mathbf{pn}}$. The $2\gcd \mathbf{n}$ minimal strips of the loops in $\lambda(A) \cup \lambda(B)$ are disjoint, so the set $h(\lambda(A) \cup \lambda(B))$ contains $2\gcd \mathbf{n}$ distinct elements. Therefore we can index the loops as in the lemma, and $h(\alpha^1) \neq h(\beta^1)$. Assume first that $h(\alpha^1) < h(\beta^1)$. Then (9) must hold, which tells us how the loops are circularly ordered. Write $\mathfrak{F} := \cup_i \operatorname{Im} \alpha^i$ and $\mathfrak{G} := \cup_i \operatorname{Im} \beta^i$. Let $\mathfrak{A} := \cup_i U_{\mathbf{td}}(\alpha^i)$ and let $\partial\mathfrak{A}$ be the boundary of $\mathfrak{A}$, and define $\mathfrak{B}$ and $\partial\mathfrak{B}$ similarly. Lemma 4.4, the definition of κ, and some suggestive reordering gives

$$\begin{aligned} |V_{\mathbf{t}}|\,(z(A,B)+\kappa(B)-\kappa(A)) = \quad & \big(\quad \operatorname{Vol}(\{\tilde{\mathbf{x}} \in C_{\mathbf{t}} : \omega_{\mathfrak{F}\cup\mathfrak{G}} \ \ (\tilde{\mathbf{x}},1) \in \mathfrak{G}\}) \\ & \quad - \operatorname{Vol}(\{\tilde{\mathbf{x}} \in \mathfrak{B} \ : \omega_{\mathfrak{G}\cup\partial\mathfrak{B}}(\tilde{\mathbf{x}},1) \in \mathfrak{G}\}) \\ & \quad - \operatorname{Vol}(\{\tilde{\mathbf{x}} \in \mathfrak{A} \ : \omega_{\mathfrak{F}\cup\partial\mathfrak{A}} \ (\tilde{\mathbf{x}},1) \in \partial\mathfrak{A}\})\big) \\ & -\big(\quad \operatorname{Vol}(\{\tilde{\mathbf{x}} \in C_{\mathbf{t}} : \omega_{\mathfrak{F}\cup\mathfrak{G}} \ \ (\tilde{\mathbf{x}},1) \in \mathfrak{F}\}) \\ & \quad - \operatorname{Vol}(\{\tilde{\mathbf{x}} \in \mathfrak{B} \ : \omega_{\mathfrak{G}\cup\partial\mathfrak{B}}(\tilde{\mathbf{x}},1) \in \partial\mathfrak{B}\}) \\ & \quad - \operatorname{Vol}(\{\tilde{\mathbf{x}} \in \mathfrak{A} \ : \omega_{\mathfrak{F}\cup\partial\mathfrak{A}} \ (\tilde{\mathbf{x}},1) \in \mathfrak{F}\})\big) \end{aligned}$$

If $\tilde{\mathbf{x}}$ is in the interior of $\mathfrak{A}$ (the boundary of this set has measure zero), then $\omega_{\mathfrak{F}\cup\mathfrak{G}}(\tilde{\mathbf{x}},1) \in \mathfrak{G}$ if and only if $\omega_{\mathfrak{F}\cup\partial\mathfrak{A}}(\tilde{\mathbf{x}},1) \in \partial\mathfrak{A}$ and $\omega_{\mathfrak{F}\cup\mathfrak{G}}(\tilde{\mathbf{x}},1) \in \mathfrak{F}$ if and only if $\omega_{\mathfrak{F}\cup\partial\mathfrak{A}}(\tilde{\mathbf{x}},1) \in \mathfrak{F}$. On the interior of $\mathfrak{B}$ we obtain similar equivalences and therefore the volume terms in the previous equation cancel on the sets $\mathfrak{A}$ and $\mathfrak{B}$; this also becomes apparent by comparing Figures 4a and 4b. Hence the two terms in the previous equation reduce to

$$\begin{aligned} |V_{\mathbf{t}}|\,(z(A,B)+\kappa(B)-\kappa(A)) = \quad & \operatorname{Vol}(\{\tilde{\mathbf{x}} \in C_{\mathbf{t}} \setminus (\mathfrak{A}\cup\mathfrak{B}) : \omega_{\mathfrak{F}\cup\mathfrak{G}}(\tilde{\mathbf{x}},1) \in \mathfrak{G}\}) \\ & - \operatorname{Vol}(\{\tilde{\mathbf{x}} \in C_{\mathbf{t}} \setminus (\mathfrak{A}\cup\mathfrak{B}) : \omega_{\mathfrak{F}\cup\mathfrak{G}}(\tilde{\mathbf{x}},1) \in \mathfrak{F}\}). \end{aligned} \tag{13}$$

The set $C_{\mathbf{t}} \setminus (\mathfrak{A}\cup\mathfrak{B})$ is the torus $C_{\mathbf{t}}$ with the $2\gcd\mathbf{n}$ disjoint minimal strips removed, so it consists of $2\gcd\mathbf{n}$ connected components, each connected component being an open strip. This set corresponds to the marked area in Figure 4c. We first inspect the open strip

$$U_{\mathbf{td},(h(\alpha^1)+r(\alpha^1)/2,h(\beta^1)-r(\beta^1)/2)},$$

which is precisely the area in between $U_{\mathbf{td}}(\alpha^1)$ and $U_{\mathbf{td}}(\beta^1)$. For any $\tilde{\mathbf{x}}$ in this set we have $\omega_{\mathfrak{F}\cup\mathfrak{G}}(\tilde{\mathbf{x}},1) \in \mathfrak{G}$ and $\omega_{\mathfrak{F}\cup\mathfrak{G}}(\tilde{\mathbf{x}},1) \notin \mathfrak{F}$. The volume of this open strip is precisely

$$\mathbf{d}_1\mathbf{t}_2(h(\beta^1) - r(\beta^1)/2 - h(\alpha^1) - r(\alpha^1)/2).$$

The contribution of this strip to (13) is precisely the volume of the strip. By repeating this procedure (13) reduces to a sum over the $2\gcd\mathbf{n}$ strips, where the open strips alternatingly contribute positively and negatively. Multiplying this sum by $|V_{\mathbf{t}}|^{-1} = (\mathbf{t}_1\mathbf{t}_2)^{-1}$ gives the right hand side of (11). This proves (11) in the case that $h(\alpha^1) < h(\beta^1)$. Both sides of (11) are antisymmetric in A and B, which proves that the formula holds in general. □

Lemma 6.3. *The process $Z + \kappa$ is a martingale.*

Proof. Fix $A \in \mathcal{S}_{\mathbf{pn}}$. We need to show that

$$\mathbb{E}\left(z(X_0, X_1) + \kappa(X_1) - \kappa(X_0) | [X_0] = A\right) = 0.$$

By the previous lemma the left side of this equation is equal to

$$\mathbb{E}\left(\left(z(X_0, X_1) + \kappa(X_1) - \kappa(X_0)\right) 1_{(X_0, X_1) \in \mathcal{P}_{\mathbf{pn}}} \middle| [X_0] = A\right).$$

If $(A, B) \in \mathcal{P}_{\mathbf{pn}}$ then $A \neq B$, so by Lemma 4.1, Statement 2, it suffices to demonstrate that

$$\sum\nolimits_{B:(A,B) \in \mathcal{P}_{\mathbf{pn}}} \left(z(A, B) + \kappa(B) - \kappa(A)\right) = 0. \tag{14}$$

For this we construct an involution τ on the set $\{B : (A, B) \in \mathcal{P}_{\mathbf{pn}}\}$ that inverts the sign of the corresponding term in (14). We first sketch what the involution τ does. For each B the map τ "rotates" each of the loops $\beta^i \in \lambda(B)$, and as a consequence of this rotation, the volumes of the "open strips" from the previous lemma immediately left and right of the minimal strip of β^i are interchanged. As in the proof of the previous lemma the summand $z(A, B) + \kappa(B) - \kappa(A)$ is a sum over the volumes over the $2 \gcd \mathbf{n}$ open strips, where the volumes of the strips alternatingly contribute positively and negatively. Since τ interchanges the volumes of the open strips, it reverses the signs of the terms corresponding to B and $\tau(B)$ in (14). Figure 4d shows how τ acts on the loops of the concerned shapes. We now construct the map τ rigorously.

Let α^i be as in the statement of Lemma 6.2. The boundary of $U_{\mathbf{td}}(\alpha^i)$ is the disjoint union of the lines $U_{\mathbf{td},\{h(\alpha^i)-r(\alpha^i)/2\}}$ and $U_{\mathbf{td},\{h(\alpha^i)+r(\alpha^i)/2\}}$. The loop α^i intersects both lines by minimality of $U_{\mathbf{td}}(\alpha^i)$. Pick $\hat{\mathbf{v}}^i \in \operatorname{Im} \alpha^i \cap U_{\mathbf{td},\{h(\alpha^i)-r(\alpha^i)/2\}}$ and $\hat{\mathbf{w}}^i \in \operatorname{Im} \alpha^i \cap U_{\mathbf{td},\{h(\alpha^i)+r(\alpha^i)/2\}}$ for all i. Then $\hat{\mathbf{v}}^i$ and $\hat{\mathbf{w}}^i$ must be in $\hat{V}_{\mathbf{t}}$ by the nature of the paths in $\mathcal{K}_{\mathbf{td}}$ and minimality of $U_{\mathbf{td}}(\alpha^i)$ (note that the loops in $\mathcal{K}_{\mathbf{td}}$ make right turns only at points in $\hat{V}_{\mathbf{t}}$). Two points $\hat{\mathbf{v}}^i$ and $\hat{\mathbf{w}}^i$ are given in Figure 4d. Equation (14) is invariant under translating the negative edges of the shape A by some $\mathbf{z} \in V_{\mathbf{t}}$, so we may assume without loss of generality that $\hat{\mathbf{v}}^1 = \hat{\mathbf{0}}$. We will now construct the involution τ that satisfies

$$z(A, \tau(B)) + \kappa(\tau(B)) - \kappa(A) = -\left(z(A, B) + \kappa(B) - \kappa(A)\right). \tag{15}$$

Pick B such that $(A, B) \in \mathcal{P}_{\mathbf{pn}}$. Let β^i be as in Lemma 6.2. We have $h(\beta^1) > h(\alpha^1)$, because we assumed that $\hat{\mathbf{v}}^1 = \hat{\mathbf{0}}$. Let us first look at β^i for some $1 \leq i < \gcd \mathbf{n}$. This loop is contained in the open strip $U_{\mathbf{td},(h(\alpha^i)+r(\alpha^i)/2, h(\alpha^{i+1})-r(\alpha^{i+1})/2)}$, and the points $\hat{\mathbf{w}}^i$ and $\hat{\mathbf{v}}^{i+1}$ are in the boundary of this open strip. Define $\bar{\beta}^i$ by

$$(\bar{\beta}^i_t)_{t \in [0,n]} := (\hat{\mathbf{w}}^i + \hat{\mathbf{v}}^{i+1} - \beta^i_{n-t})_{t \in [0,n]},$$

where n is the length of the loop β^i. Intuitively, $\bar{\beta}^i$ is obtained by rotating the path β^i over an angle π around the point half way in between $\hat{\mathbf{w}}^i$ and $\hat{\mathbf{v}}^{i+1}$, and inverting the direction in which we trace the path so obtained. It is straightforward to prove that $\bar{\beta}^i \in \mathcal{K}_{\mathbf{td}}$. Observe that

$$\begin{aligned} U_{\mathbf{td}}(\bar{\beta}^i) &= \hat{\mathbf{w}}^i + \hat{\mathbf{v}}^{i+1} - U_{\mathbf{td}}(\beta^i) \\ &= U_{\mathbf{td},\{h(\alpha^i)+r(\alpha^i)/2\}} + U_{\mathbf{td},\{h(\alpha^{i+1})-r(\alpha^{i+1})/2\}} - U_{\mathbf{td},[h(\beta^i)-r(\beta^i)/2,h(\beta^i)+r(\beta^i)/2]} \\ &= U_{\mathbf{td},h(\alpha^i)+r(\alpha^i)/2+h(\alpha^{i+1})-r(\alpha^{i+1})/2-h(\beta^i)+[-r(\beta^i)/2,r(\beta^i)/2]}. \end{aligned} \tag{16}$$

Define equivalently $\bar{\beta}^{\gcd \mathbf{n}} \in \mathcal{K}_{\mathbf{td}}$ by

$$(\bar{\beta}^{\gcd \mathbf{n}}_t)_{t\in[0,n]} := (\hat{\mathbf{w}}^{\gcd \mathbf{n}} + \hat{\mathbf{v}}^1 - \beta^{\gcd \mathbf{n}}_{n-t})_{t\in[0,n]},$$

and

$$U_{\mathbf{td}}(\bar{\beta}^{\gcd \mathbf{n}}) = U_{\mathbf{td},h(\alpha^{\gcd \mathbf{n}})+r(\alpha^{\gcd \mathbf{n}})/2+h(\alpha^1)+\frac{\mathbf{t}_1}{\mathbf{d}_1}-r(\alpha^1)/2-h(\beta^{\gcd \mathbf{n}})+[-r(\beta^{\gcd \mathbf{n}})/2,r(\beta^{\gcd \mathbf{n}})/2]}. \tag{17}$$

The loops $\bar{\beta}^i$ are simple because the loops β^i are simple. For each i, the loops β^i and $\bar{\beta}^i$ are contained in the same connected component of $C_{\mathbf{t}} \setminus \cup_j U_{\mathbf{td}}(\alpha^j)$. Therefore the loops $\bar{\beta}^i$ are disjoint. By Lemma 3.2 there exists a shape $\bar{B}$ with $\lambda(\bar{B}) = \{\bar{\beta}^i : 1 \le i \le \gcd \mathbf{n}\}$. The $2 \gcd \mathbf{n}$ loops in $\lambda(A) \cup \lambda(\bar{B})$ are contained in $2 \gcd \mathbf{n}$ disjoint strips. Moreover

$$h(\alpha^1) < h(\bar{\beta}^1) < ... < h(\alpha^{\gcd \mathbf{n}}) < h(\bar{\beta}^{\gcd \mathbf{n}}),$$

so that A and $\bar{B}$ are neighbours by Lemma 5.8. Hence $(A, \bar{B}) \in \mathcal{P}_{\mathbf{pn}}$. Define $\tau : B \mapsto \bar{B}$. To see that τ is an involution, simply observe that the loops corresponding to $\tau(\tau(B))$ are exactly the loops β^i. It suffices to prove (15). Note that both sides of (15) are given by (11), and that $h(\alpha^1) < h(\beta^1)$ and $h(\alpha^1) < h(\bar{\beta}^1)$. The numbers $h(\bar{\beta}^i)$ and $r(\bar{\beta}^i)$ are given by (16) and (17). In particular, $r(\bar{\beta}^i) = r(\beta^i)$ for all i,

$$\begin{aligned} h(\bar{\beta}^i) \quad &= h(\alpha^i) + r(\alpha^i)/2 + h(\alpha^{i+1}) - r(\alpha^{i+1})/2 - h(\beta^i) \quad \text{for } 1 \le i < \gcd \mathbf{n} \text{ and} \\ h(\bar{\beta}^{\gcd \mathbf{n}}) &= h(\alpha^{\gcd \mathbf{n}}) + r(\alpha^{\gcd \mathbf{n}})/2 + h(\alpha^1) + \frac{\mathbf{t}_1}{\mathbf{d}_1} - r(\alpha^1)/2 - h(\beta^{\gcd \mathbf{n}}). \end{aligned}$$

Substituting these numbers into (11) indeed gives (15). □

7. Proof of Theorem 1.2

Recall that $\mathbb{P}_{\mathbf{pn}}$ is the probability measure for the walk $X^{\mathbf{pn}}$, that the distribution of $[X]$ is invariant in $\mathbb{P}_{\mathbf{pn}}$, and that $\tilde{\mathbb{P}}_{\mathbf{pn}}$ is the uniform probability measure on $\mathcal{K}_{\mathbf{td}}^{\gcd \mathbf{n}} \times \mathcal{K}_{\mathbf{td}}^{\gcd \mathbf{n}}$. Recall also that we defined

$$\begin{aligned} \mathcal{D}_{\mathbf{tn}} &= \{\text{the } 2 \gcd \mathbf{n} \text{ minimal strips of } (\boldsymbol{\alpha}, \boldsymbol{\beta}) \text{ are disjoint}\} \qquad \subset \mathcal{K}_{\mathbf{td}}^{\gcd \mathbf{n}} \times \mathcal{K}_{\mathbf{td}}^{\gcd \mathbf{n}}, \\ \mathcal{P}_{\mathbf{pn}} &= \{\text{the } 2 \gcd \mathbf{n} \text{ minimal strips of } \lambda(A) \text{ and } \lambda(B) \text{ are disjoint}\} \subset \mathcal{S}_{\mathbf{pn}} \times \mathcal{S}_{\mathbf{pn}}, \end{aligned}$$

and define

$$\mathcal{M}_{\mathbf{pn}} := \{(A,B) \in \mathcal{S}_{\mathbf{pn}} \times \mathcal{S}_{\mathbf{pn}} : A \sim B\}.$$

Define the two measures ${}_*\mathbb{P}$ and ${}_*\tilde{\mathbb{P}}$ on $\mathcal{M}_{\mathbf{pn}}$ by

$${}_*\mathbb{P}(\{(A,B)\}) := \mathbb{P}([X_0]=A,[X_1]=B) \quad = \begin{cases} 1/(|\mathcal{M}_{\mathbf{pn}}|+|\mathcal{S}_{\mathbf{pn}}|) & \text{if } A \neq B \\ 2/(|\mathcal{M}_{\mathbf{pn}}|+|\mathcal{S}_{\mathbf{pn}}|) & \text{if } A = B \end{cases}, \tag{18}$$

$${}_*\tilde{\mathbb{P}}(\{(A,B)\}) := \tilde{\mathbb{P}}(\lambda(A)=\boldsymbol{\alpha},\lambda(B)=\boldsymbol{\beta}) = (\gcd \mathbf{n})!^2/|\mathcal{K}_{\mathbf{td}}|^{2\gcd \mathbf{n}}, \tag{19}$$

where (18) follows from Lemma 4.1, Statement 2. The factor $(\gcd \mathbf{n})!^2$ in (19) comes from the $(\gcd \mathbf{n})!$ ways in which we can order the loops in $\boldsymbol{\alpha}$ and the $(\gcd \mathbf{n})!$ ways in which we can order the loops $\boldsymbol{\beta}$. We remark that ${}_*\mathbb{P}$ is a probability measure and that the measure ${}_*\tilde{\mathbb{P}}$ is uniform. Define c to be the normalising constant of ${}_*\tilde{\mathbb{P}}$, so that $c_*\tilde{\mathbb{P}}$ is the uniform probability measure on $\mathcal{M}_{\mathbf{pn}}$.

Lemma 7.1. *The following statements hold true as $\mathbf{p} \to \infty$:*

1. $c_{\mathbf{pn}} \to \frac{1}{2}\binom{2\gcd \mathbf{n}}{\gcd \mathbf{n}}$,
2. *The total variation distance between ${}_*\mathbb{P}_{\mathbf{pn}}$ and $c_{\mathbf{pn}*}\tilde{\mathbb{P}}_{\mathbf{pn}}$ goes to zero,*
3. $\mathbb{P}_{\mathbf{pn}}([X_0^{\mathbf{pn}}]=[X_1^{\mathbf{pn}}])$, $\mathbb{P}_{\mathbf{pn}}(([X_0^{\mathbf{pn}}],[X_1^{\mathbf{pn}}]) \notin \mathcal{P}_{\mathbf{pn}}) \to 0$,
4. $\sigma^2(\hat{X}^{\mathbf{pn}}) - \sigma^2(Y^{\mathbf{pn}})$, $\sigma^2(Y^{\mathbf{pn}}) - \sigma^2(Z^{\mathbf{pn}}) \to 0$.

Proof. Note that

$${}_*\tilde{\mathbb{P}}_{\mathbf{pn}}(\mathcal{M}_{\mathbf{pn}}) = \tilde{\mathbb{P}}_{\mathbf{pn}}(\text{there exist shapes } A \sim B \text{ with } \lambda(A)=\boldsymbol{\alpha} \text{ and } \lambda(B)=\boldsymbol{\beta}).$$

The event $\mathcal{D}_{\mathbf{tn}}$ has high $\tilde{\mathbb{P}}_{\mathbf{pn}}$-probability by Lemma 5.9, so we condition on this event. Lemma 5.8 tells us that, conditional on $\mathcal{D}_{\mathbf{tn}}$, the inclusion

$$(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \{\text{there exist shapes } A \sim B \text{ with } \lambda(A)=\boldsymbol{\alpha} \text{ and } \lambda(B)=\boldsymbol{\beta}\}$$

depends only on the ordering of the numbers $h(\alpha^1),...,h(\alpha^{\gcd \mathbf{n}}),h(\beta^1),...,h(\beta^{\gcd \mathbf{n}})$. Conditional on $\mathcal{D}_{\mathbf{tn}}$ the ordering of these numbers is uniformly random, and the probability that a random ordering is intertwining as in Lemma 5.8 is precisely $2\binom{2\gcd \mathbf{n}}{\gcd \mathbf{n}}^{-1}$. This is Statement 1. By (18) the total variation distance between ${}_*\mathbb{P}_{\mathbf{pn}}$ and the uniform probability measure $c_{\mathbf{pn}*}\tilde{\mathbb{P}}_{\mathbf{pn}}$ is

$${}_*\mathbb{P}_{\mathbf{pn}}(A=B) - c_{\mathbf{pn}*}\tilde{\mathbb{P}}_{\mathbf{pn}}(A=B) \le c_{\mathbf{pn}*}\tilde{\mathbb{P}}_{\mathbf{pn}}(A=B) \le c_{\mathbf{pn}}(1-\tilde{\mathbb{P}}_{\mathbf{pn}}(\mathcal{D}_{\mathbf{tn}})) \to_{\mathbf{p}\to\infty} 0.$$

For the third statement we choose to work in the measure $c_{\mathbf{pn}*}\tilde{\mathbb{P}}_{\mathbf{pn}}$, so that

$$c_{\mathbf{pn}*}\tilde{\mathbb{P}}_{\mathbf{pn}}(A=B) \le c_{\mathbf{pn}*}\tilde{\mathbb{P}}_{\mathbf{pn}}((A,B) \notin \mathcal{P}_{\mathbf{pn}}) \le c_{\mathbf{pn}}(1-\tilde{\mathbb{P}}_{\mathbf{pn}}(\mathcal{D}_{\mathbf{tn}})) \to_{\mathbf{p}\to\infty} 0.$$

Finally we prove the fourth statement. For the proof of the first assertion we refer to Statement 3 and (10). For the proof of the second assertion we recall that $D = Y - Z$, so that it suffices to demonstrate that the diffusivity of $D^{\mathbf{pn}}$ goes to zero as $\mathbf{p} \to \infty$. Recall that $d = y - z$ so that by definition of y and z,

$$d(A,B) = \begin{cases} 0 & \text{if } (A,B) \in \mathcal{P}_{\mathbf{pn}} \\ \hat{g} - \hat{f} + \kappa(B) - \kappa(A) & \text{if } (A,B) \notin \mathcal{P}_{\mathbf{pn}}, A \sim B \text{ and } A \neq B \ , \\ \kappa(B) - \kappa(A) & \text{otherwise} \end{cases}$$

where f and g are chosen to be neighbours with shapes A and B respectively. Observe that d equals zero on $\mathcal{P}_{\mathbf{pn}}$ and that $|d(A,B)| \leq |\hat{g} - \hat{f}| + |\kappa(B)| + |\kappa(A)| \leq 3$. By Theorem 6.1,

$$\sigma^2(D^{\mathbf{pn}}) \leq \mathbb{E}_{\mathbf{pn}}(d(X_0^{\mathbf{pn}}, X_1^{\mathbf{pn}})^2) \leq 9\mathbb{P}_{\mathbf{pn}}(([X_0^{\mathbf{pn}}],[X_1^{\mathbf{pn}}]) \notin \mathcal{P}_{\mathbf{pn}}).$$

By Statement 3 the upper bound goes to zero as $\mathbf{p} \to \infty$. □

We make some remarks before finishing the proof of the main result. In the past sections we made some modifications to the original process which have vanishing errors as $\mathbf{p} \to \infty$. Instead of sampling a random neighbour of a height function, we sample random loops from the space $\mathcal{K}_{\mathbf{td}}$, a space that is much better understood. A randomly sampled loop $\alpha \in \mathcal{K}_{\mathbf{td}}$ is close to a diagonal line, and its relevant characteristics are determined solely by the number $h(\alpha)$. This effectively reduces the original model to a one-dimensional model (since $(\mathbf{d}_1/\mathbf{t}_1)h(\alpha)$ takes values in the unit interval). Furthermore, the distribution of the random variable $(\mathbf{d}_1/\mathbf{t}_1)h(\alpha)$ in the uniform measure on $\mathcal{K}_{\mathbf{td}}$ converges to the continuous uniform distribution (on the unit interval). Therefore, we further reduce from a discrete model with a difficult combinatorial structure to a continuous model. On this continuous one-dimensional model we perform our final calculations.

Proof of Theorem 1.2. Write ${}_*\tilde{\mathbb{E}}_{\mathbf{pn}}$ for the expectation of a random variable under the probability measure $c_{\mathbf{pn}*}\tilde{\mathbb{P}}_{\mathbf{pn}}$. By the previous lemma we have

$$\begin{aligned} \lim_{\mathbf{p}} \sigma^2(\hat{X}^{\mathbf{pn}}) &= \lim_{\mathbf{p}} \mathbb{E}_{\mathbf{pn}}\left((z(X_0,X_1) + \kappa_1 - \kappa_0)^2\right) \\ &= \lim_{\mathbf{p}} {}_*\mathbb{E}_{\mathbf{pn}}\left((z(A,B) + \kappa(B) - \kappa(A))^2\right) \\ &= \lim_{\mathbf{p}} {}_*\tilde{\mathbb{E}}_{\mathbf{pn}}\left((z(A,B) + \kappa(B) - \kappa(A))^2\right) \\ &= \lim_{\mathbf{p}} {}_*\tilde{\mathbb{E}}_{\mathbf{pn}}\left((z(A,B) + \kappa(B) - \kappa(A))^2 \middle| \mathcal{P}_{\mathbf{pn}}\right). \end{aligned} \tag{20}$$

Note that we are allowed to change measure and condition on a high-probability event because the integrand within the expectations is uniformly bounded. We want to calculate this final expectation in the probability measure $\tilde{\mathbb{P}}$. Write A and B for the maps

$$\mathcal{D}_{\mathbf{tn}} \to \mathcal{S}_{\mathbf{pn}},\ (\boldsymbol{\alpha},\boldsymbol{\beta}) \mapsto \lambda^{-1}(\boldsymbol{\alpha}) \ \text{ and } \ \mathcal{D}_{\mathbf{tn}} \to \mathcal{S}_{\mathbf{pn}},\ (\boldsymbol{\alpha},\boldsymbol{\beta}) \mapsto \lambda^{-1}(\boldsymbol{\beta})$$

respectively, which are random variables in the restriction of $\tilde{\mathbb{P}}_{\mathbf{pn}}$ to $\mathcal{D}_{\mathbf{tn}}$. The final expectation in (20) equals

$$\tilde{\mathbb{E}}_{\mathbf{pn}}\left((z(A,B)+\kappa(B)-\kappa(A))^2\middle|\mathcal{D}_{\mathbf{tn}}, A\sim B\right). \tag{21}$$

Write $\mathcal{O}$ for the event

$$\{h(\alpha^i)<h(\alpha^j) \text{ and } h(\beta^i)<h(\beta^j) \text{ for all } i<j\}\subset \mathcal{K}_{\mathbf{td}}^{\gcd\mathbf{n}}\times\mathcal{K}_{\mathbf{td}}^{\gcd\mathbf{n}}.$$

The event $\{A\sim B\}$ and the random variable $(z(A,B)+\kappa(B)-\kappa(A))^2$ are, conditional on $\mathcal{D}_{\mathbf{tn}}$, independent of $\mathcal{O}$. This is because both are invariant under permuting the loops in $\boldsymbol{\alpha}$ or permuting the loops in $\boldsymbol{\beta}$. Therefore (21) equals

$$\tilde{\mathbb{E}}_{\mathbf{pn}}\left((z(A,B)+\kappa(B)-\kappa(A))^2\middle|\mathcal{D}_{\mathbf{tn}}, A\sim B,\mathcal{O}\right). \tag{22}$$

Write $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ for the events

$$\{h(\alpha^1)<h(\beta^1)<...<h(\alpha^{\gcd\mathbf{n}})<h(\beta^{\gcd\mathbf{n}})\},$$
$$\{h(\beta^1)<h(\alpha^1)<...<h(\beta^{\gcd\mathbf{n}})<h(\alpha^{\gcd\mathbf{n}})\}$$

respectively. These events are disjoint, and by Lemma 5.8

$$\mathcal{D}_{\mathbf{tn}}\cap\{A\sim B\}\cap\mathcal{O}=\mathcal{D}_{\mathbf{tn}}\cap(\mathcal{O}_\alpha\cup\mathcal{O}_\beta).$$

Note that in (22) both the integrand and the three conditioning events are invariant under interchanging $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, and therefore (22) equals

$$\tilde{\mathbb{E}}_{\mathbf{pn}}((z(A,B)+\kappa(B)-\kappa(A))^2|\mathcal{D}_{\mathbf{tn}},\mathcal{O}_\alpha).$$

By Lemma 6.2 this equals

$$\tilde{\mathbb{E}}_{\mathbf{pn}}\left(\left(1-\textstyle\sum_i\left(2\frac{\mathbf{d}_1}{\mathbf{t}_1}h(\beta^i)-2\frac{\mathbf{d}_1}{\mathbf{t}_1}h(\alpha^i)\right)\right)^2\middle|\mathcal{D}_{\mathbf{tn}},\mathcal{O}_\alpha\right).$$

As $\mathbf{p}\to\infty$ the probability of the event $\mathcal{O}_\alpha$ remains uniformly positive. Therefore this expectation equals, in the limit in $\mathbf{p}$,

$$\lim_{\mathbf{p}}\sigma^2(\hat{X}^{\mathbf{pn}})=\lim_{\mathbf{p}}\tilde{\mathbb{E}}_{\mathbf{pn}}\left(\left(1-\textstyle\sum_i\left(2\frac{\mathbf{d}_1}{\mathbf{t}_1}h(\beta^i)-2\frac{\mathbf{d}_1}{\mathbf{t}_1}h(\alpha^i)\right)\right)^2\middle|\mathcal{O}_\alpha\right),$$

where we no longer condition on $\mathcal{D}_{\mathbf{tn}}$. By Lemma 5.5 the distribution of

$$\frac{\mathbf{d}_1}{\mathbf{t}_1}(h(\alpha^1),...,h(\alpha^{\gcd\mathbf{n}}),h(\beta^1),...,h(\beta^{\gcd\mathbf{n}}))$$

converges to the uniform continuous distribution on $[0,1]^{2\gcd\mathbf{n}}$. Suppose that I and J are independent random variables having the uniform continuous distribution on $[0,1]^{\gcd\mathbf{n}}$ in some probability measure $\mathbb{P}$. Then

$$\lim_{\mathbf{p}}\sigma^2(\hat{X}^{\mathbf{pn}})=\mathbb{E}\left(\left(1-\textstyle\sum_i(2J_i-2I_i)\right)^2\middle|I_1<J_1<...<I_{\gcd\mathbf{n}}<J_{\gcd\mathbf{n}}\right).$$

The conditioning event has probability $(2\gcd \mathbf{n})!^{-1}$. We observe that

$$
\begin{aligned}
&\mathbb{E}\left(\left(1-\sum_i(2J_i-2I_i)\right)^2 1_{I_1<J_1<\ldots<I_{\gcd \mathbf{n}}<J_{\gcd \mathbf{n}}}\right)\\
&=\int_{[0,1]^{2\gcd \mathbf{n}}}\left(1-\sum_i(2J_i-2I_i)\right)^2 1_{I_1<J_1<\ldots<I_{\gcd \mathbf{n}}<J_{\gcd \mathbf{n}}} d(I,J)\\
&=\int_{\Delta^{2\gcd \mathbf{n}}}\left(\sum_{j=0}^{2\gcd \mathbf{n}}(-1)^j x_j\right)^2 dx\\
&=(1+2\gcd \mathbf{n})!^{-1}.
\end{aligned}
$$

For the second equality we perform the change of variables $I_i=\sum_{j=0}^{2i-2}x_j$, $J_i=\sum_{j=0}^{2i-1}x_j$ and $1=\sum_{j=0}^{2\gcd \mathbf{n}}x_j$. We then integrate over the unit simplex, precisely the set where the indicator, which now depends on x, is positive. For the third equality we gather terms of equal powers and express the integral in terms of the multivariate beta function. We conclude that

$$
\lim_{\mathbf{p}\to\infty}\sigma^2(\hat{X}^{\mathbf{pn}})=\frac{(1+2\gcd \mathbf{n})!^{-1}}{(2\gcd \mathbf{n})!^{-1}}=(1+2\gcd \mathbf{n})^{-1},
$$

which is Theorem 1.2. □

Acknowledgement

I want to thank James Norris for supervising the writing of this paper. He inspired me to study height functions, leading to the result presented here. His comments have greatly helped me in writing the paper.

This work was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) grant EP/L016516/1 for the University of Cambridge Centre for Doctoral Training, the Cambridge Centre for Analysis.

References

[1] Emmanuel Boissard, Serge Cohen, Thibault Espinasse, and James Norris. Diffusivity of a random walk on random walks. *Random Structures & Algorithms*, 47(2):267–283, 2015.

[2] Thibault Espinasse, Nadine Guillotin-Plantard, and Philippe Nadeau. A combinatorial approach to a model of constrained random walkers. *Combinatorics, Probability and Computing*, 25:222–235, 3 2016.

[3] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.